ANALYTIC GEOMETRY

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PREFACE

The purpose of this book is to present in a short course those parts of Analytic Geometry which are essential for the study of Calculus. The material has been so arranged that topics which are less important may be omitted without a loss of continuity, and the text is therefore adapted for use in classes which aim to cover in one year the fundamental principles and applications of both Analytic Geometry and Calculus. For such classes Chapters I–VI, VIII, and the earlier sections of Chapters VII and IX form a course which can be completed in about thirty-five lessons. In order to provide material for study if more time is available, a discussion of the general equation of the second degree and short chapters on Tangents and Normals and Solid Analytic Geometry have been included.

In the preparation of the book the authors have used as a foundation a pamphlet written by Professor William Beebe for use in his classes. The authors wish to express their thanks to Dr. David D. Leib and to Dr. Levi L. Conant for many valuable criticisms and suggestions.

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INTRODUCTION

DEFINITIONS AND FORMULAS FROM ALGEBRA AND TRIGONOMETRY

1. The Quadratic. — An expression reducible to the form

$$Ax^2 + Bx + C,$$

where A, B, and C are any constants, is called a quadratic in x.

2. Solution of the Quadratic. — The roots of any quadratic equation may be found by completing the square, or by using the quadratic formula.

(*a*) To solve by completing the square, transpose the constant term to the right-hand member and divide both sides of the equation by the coefficient of x^2. In the resulting equation add to both members the square of half the coefficient of x and extract the square root.

(*b*) If the above rule is applied to the general quadratic $Ax^2 + Bx + C = 0$, we get

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

which is known as the *quadratic formula*. To solve an equation by means of this formula simply substitute for A, B, and C in the formula their values in the given equation.

3. The Discriminant. — The expression under the radical, $B^2 - 4AC$, is called the *discriminant* of the quadratic. It

is denoted by the Greek letter Δ. The quadratic formula shows that the roots of a quadratic equation are

$$\begin{array}{ll} \text{imaginary,} & \text{if } \Delta < 0; \\ \text{real and equal,} & \text{if } \Delta = 0; \\ \text{real and unequal,} & \text{if } \Delta > 0. \end{array}$$

4. Logarithms. — The logarithm of a number to a given base is that exponent by which the base must be affected in order that the result may be equal to the number. Thus if $b^x = N$, we say x is the logarithm of N to the base b and write $x = \log_b N$.

The base b may be any positive number different from 0 and 1. Two systems of logarithms are in common use: the natural, or Naperian, where the base is $e = 2.71828\ldots$; and the common, or Briggs, where the base is 10.

Important properties of logarithms to any base are the following:

$$\log_b 1 = 0; \qquad \log_b b = 1;$$

$$\log_b MN = \log_b M + \log_b N; \qquad \log_b \frac{M}{N} = \log_b M - \log_b N;$$

$$\log_b N^k = k \log_b N; \qquad \log_b \sqrt[k]{N} = \frac{1}{k} \log_b N;$$

$$\log_c N = \frac{\log_b N}{\log_b c} = \log_b N \cdot \log_c b.$$

5. Angles. — In trigonometry an angle is supposed to be generated by revolving a line, called the *generating line*, about the vertex from one side of the angle, called the *initial line*, to the other side, called the *terminal line*. If the rotation is counter-clockwise, the angle is called positive; if clockwise, it is called negative.

Degree Measure. — The ordinary unit of angle measurement is one ninetieth part of a right angle, which is called a *degree*. One sixtieth part of a degree is called a minute and one sixtieth part of a minute is called a second. Thus

$60'' =$ one minute;
$60' =$ one degree;
$360° =$ one complete revolution $=$ four right angles.

Circular Measure. — The unit of circular measure is a central angle subtended by an arc equal to the radius of the circle. It is called a *radian*. We find at once that

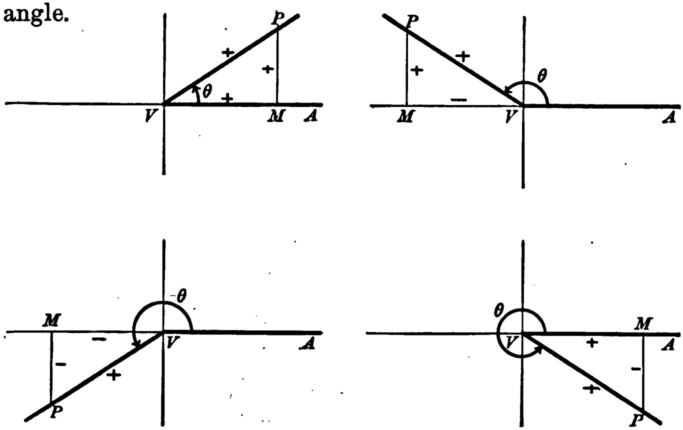
$$\pi \text{ radians} = 180°;$$

$$1 \text{ radian} = \frac{180°}{\pi};$$

$$1° = \frac{\pi}{180} \text{ radians.}$$

From this definition we have the important result that in any circle the length of an arc equals the product of the measure of its subtended central angle in radians and the length of the radius.

6. Trigonometric Functions. — From any point P of the terminal line VP of the angle $\theta = AVP$ drop a perpendicular on the initial line of the angle (produced if necessary). The ratios of the sides of the triangle thus formed are called the trigonometric ratios or functions of the given angle.



They are

$$\sin\theta = \frac{MP}{VP};\qquad \cos\theta = \frac{VM}{VP};\qquad \tan\theta = \frac{MP}{VM};$$

$$\csc\theta = \frac{VP}{MP};\qquad \sec\theta = \frac{VP}{VM};\qquad \cot\theta = \frac{VM}{MP}.$$

In these ratios the side of the triangle opposite the angle is considered positive when it extends up from the initial line, negative when down; the adjacent side is considered positive if it extends to the right of the vertex, negative if to the left; the hypothenuse is always positive.

7. Relations between the Functions. — For any angle,

$$\sec\theta = \frac{1}{\cos\theta};\qquad \cot\theta = \frac{\cos\theta}{\sin\theta};$$

$$\csc\theta = \frac{1}{\sin\theta};\qquad \sin^2\theta + \cos^2\theta = 1;$$

$$\cot\theta = \frac{1}{\tan\theta};\qquad 1 + \tan^2\theta = \sec^2\theta;$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta};\qquad 1 + \cot^2\theta = \csc^2\theta.$$

8. Reduction of Angles. —

$$\sin(-\theta) = -\sin\theta;\qquad \sin\left(\frac{\pi}{2}\pm\theta\right) = \cos\theta;$$

$$\cos(-\theta) = \cos\theta;\qquad \cos\left(\frac{\pi}{2}\pm\theta\right) = \mp\sin\theta;$$

$$\tan(-\theta) = -\tan\theta;\qquad \tan\left(\frac{\pi}{2}\pm\theta\right) = \mp\cot\theta;$$

$$\sin(\pi\pm\theta) = \mp\sin\theta;$$
$$\cos(\pi\pm\theta) = -\cos\theta;$$
$$\tan(\pi\pm\theta) = \pm\tan\theta.$$

9. Special Angles. —

Angle	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0	∞	0

10. Formulas for the Sum and Difference of Two Angles. —

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi;$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi;$$

$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}.$$

11. Double and Half Angle Formulas. —

$$\sin 2\theta = 2\sin\theta\cos\theta;$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1;$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta};$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}; \quad \cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}};$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}.$$

12. Triangle Formulas. — In any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{(Law of Sines);}$$

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \text{(Law of Cosines).}$$

13. The Greek Alphabet —

Letters	Names	Letters	Names
A, α	Alpha	N, ν	Nu
B, β	Beta	Ξ, ξ	Xi
Γ, γ	Gamma	O, o	Omicron
Δ, δ	Delta	Π, π	Pi
E, ϵ	Epsilon	P, ρ	Rho
Z, ζ	Zeta	Σ, σ	Sigma
H, η	Eta	T, τ	Tau
Θ, θ	Theta	Υ, υ	Upsilon
I, ι	Iota	Φ, ϕ	Phi
K, κ	Kappa	X, χ	Chi
Λ, λ	Lambda	Ψ, ψ	Psi
M, μ	Mu	Ω, ω	Omega

CHAPTER I

COÖRDINATES AND EQUATIONS

1. Introduction. — The chief feature of Analytic Geometry, which distinguishes it from the geometry which the student has hitherto studied, is its extensive use of algebraic methods in the solution of geometric problems. Just as the use of symbols in algebra makes possible the ready solution of many problems which would be difficult if not impossible by the processes of arithmetic, so the use of algebraic reasoning simplifies much of geometry and widens its scope. The student is already familiar with some of these applications; for example, the theorems stating the numerical properties of a triangle are derived in whole or part by algebraic reasoning.

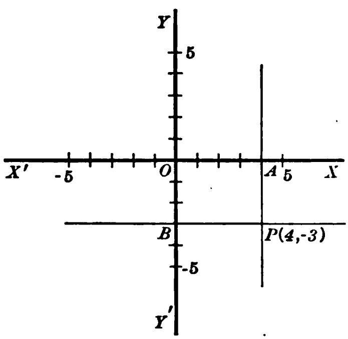
In trigonometry also the greater part of the reasoning is of the same character, and further simplification is gained by the introduction of negative numbers. For example, the law of cosines, $a^2 = b^2 + c^2 - 2bc \cos A$, states in compact language the contents of three theorems of plane geometry, since cosine A is positive or negative according as angle A is acute or obtuse. Other examples of a similar character may occur to the student.

2. Position; Cartesian Coördinates. — The most important help in the extension of geometric analysis has been found in the location of points in a plane or in space by Cartesian coördinates, an adaptation to mathematics of the means of expressing position used in ordinary life.

We ordinarily locate a point by stating its direction and distance from a fixed reference point, as — Philadelphia is

ninety miles southwest of New York. Less commonly, its position is fixed by its distances from a pair of fixed perpendicular lines. An illustration of this is found in the familiar direction given in a city to go, for example, eight blocks east and three north to reach a desired destination. Both of these methods are used analytically; the second is simpler and will be discussed first.

In the figure the reference lines are the perpendiculars $X'X$ and $Y'Y$. These are called the **axes of coördinates** or **coördinate axes**, and their intersection O is called the **origin**. The two axes divide the plane into four quadrants numbered as in trigonometry. The position of a point P is determined by measuring its distance from $Y'Y$ along a parallel to $X'X$, and its distance from $X'X$ along a parallel to $Y'Y$.



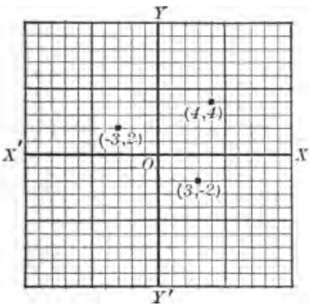
Distances measured to the right of $Y'Y$ or up from $X'X$ are called *positive*, those to the left or down *negative*. The measures of these distances, with the proper signs prefixed, are called the **coördinates** of the point; the one measured along a line parallel to $X'X$ is the **x-coördinate** or **abscissa**; the one parallel to $Y'Y$ is the **y-coördinate** or **ordinate**. Thus in the figure the abscissa of P is $BP = +4$, its ordinate is $AP = -3$. The coördinate axis $X'X$ is called the *axis of abscissas* or **x-axis**, and the axis $Y'Y$ the *axis of ordinates* or **y-axis**.

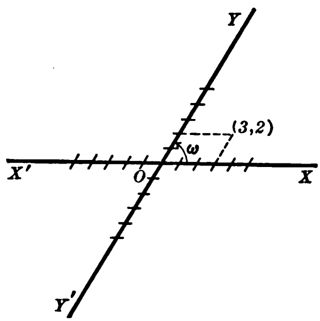
3. Notation. — In naming a point by its coördinates, write them in a parenthesis, putting the abscissa first. Thus in the figure P is the point $(4, -3)$. In case the coördinates

are variable or unknown, the abscissa is denoted by the letter x, and the ordinate by y. Fixed points of which the coördinates are not known or are arbitrary, will be distinguished by means of subscripts, being lettered P_1, P_2, etc., and represented by the coördinates (x_1, y_1), (x_2, y_2), etc.

4. Plotting of Points. — It will be seen that any point in a plane is fixed by means of its coördinates; for the abscissa locates it on a parallel to the y-axis and the ordinate on a parallel to the x-axis, and these meet in one point.

To locate a point whose coördinates are given (or as commonly stated, to *plot* the point), first choose a convenient unit of measure, then measure off the abscissa on the x-axis





and the ordinate from the end of the abscissa. Thus to plot $(-3, 2)$ count 3 units to the left on the x-axis and 2 units up. This is especially convenient when using coördinate paper, *i.e.* paper ruled with sets of parallel lines.

The system of coördinates which has been described is the rectangular system and is a particular case of Cartesian coördinates. In the general Cartesian system the axes are not necessarily perpendicular, but may be *oblique*. All of the above definitions hold for the oblique system, which is illustrated above.

PROBLEMS

1. Plot accurately the points:

(a) $(2, 5)$, $(3, 0)$, $(-\sqrt{2}, 5)$;
(b) $(-3, 0)$, $(5, -8)$, $(1, \sqrt{12})$;
(c) $(2, -6)$, $(-3, -3)$, $(0, -\sqrt{7})$;
(d) $(-1, 2)$, $(-2, -3)$, $(-\sqrt{3}, 0)$.

2. Draw the triangle whose vertices are as follows:

(a) $(3, -1)$, $(-2, 5)$, $(-8, -4)$;
(b) $(2, 4)$, $(4, \sqrt{8})$, $(3, 11)$;
(c) $(5, 3)$, $(5 + 4\sqrt{2}, 3 + 4\sqrt{2})$, $(5 + 8\sqrt{2}, 3 - 8\sqrt{2})$.

3. Draw the quadrilateral whose vertices are as follows:

(a) $(-2, 0)$, $(2, 4)$, $(6, 0)$, $(2, -4)$;
(b) $(0, -6)$, $(8, 2)$, $(0, 6)$, $(-8, 2)$.

4. Construct with compasses on coördinate paper $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and estimate the value of each.

Hint. — Each radical will be the measure of the hypothenuse or leg of a right triangle.

5. Draw a circle of radius 10 with its center at the origin. Draw radii at 20° intervals up to 180°. Estimate from the figure the coördinates of the ends of these radii.

6. From the figure of problem 5 make a table of sines and cosines of the angles drawn.

7. What is the abscissa of all points on the y-axis? the ordinate of all points on the x-axis? What are the coördinates of the origin?

8. To what quadrants is a point limited if its abscissa and ordinate have like signs? unlike signs?

9. If a point moves on a parallel to the x-axis, which of its coördinates remains constant? which if it moves on a perpendicular to the x-axis?

10. (a) What is the locus of a point whose abscissa is 6? whose ordinate is -6?

(b) What is the locus of all points having the same abscissa? having the same ordinate?

11. What is the locus of points whose abscissas and ordinates are (a) equal? (b) numerically equal, but of unlike sign? Why?

12. A square whose side is $2a$ has its center at the origin and its sides parallel to the axes. Find the coördinates of its vertices.

13. An equilateral triangle of side a has one vertex at the origin and the opposite side parallel to the y-axis. Find the coördinates of the other vertices. *Ans.* $\left(\frac{a}{2}\sqrt{3},\ \pm\frac{a}{2}\right)$, or $\left(-\frac{a}{2}\sqrt{3},\ \pm\frac{a}{2}\right)$.

14. A rhombus has one angle of 60° and two vertices at (0, 0) and (a, 0). Find the coördinates of the other vertices if (a) both are in the first quadrant; (b) one is in the second quadrant.

Ans. (a): $\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$, $\left(\frac{3a}{2}, \frac{a\sqrt{3}}{2}\right)$.

15. A regular hexagon of side a is placed so that one diagonal lies along the x-axis and the center is at the origin. Find the coördinates of the vertices.

5. Directed Lines. — We have referred to the advantage of using positive and negative lines in trigonometry and have defined the signs of coördinates. In analytic geometry we constantly use *directed lines*, that is, lines whose lengths are reckoned as positive or negative according to the direction in which they are read. For example, if the positive direction is from left to right, and AB is 8 units long, then

$$A \longrightarrow B$$

$AB = 8$, while $BA = -8$. Thus, *changing the direction of reading a line changes its sign*, i.e. $BA = -AB$.

In adding or subtracting line segments great care should be taken to avoid errors of sign. It is advisable for beginners to read all segments in one direction in performing such operations, later reversing segments as required. For example, let us find the relation between AB, BC, and AC in the figure.

$$A \qquad\qquad C \qquad\qquad B$$

We have $$AB = AC + CB = AC - BC.$$

In the Cartesian coördinate system *the positive direction on all lines parallel to the x-axis is to the right; on lines not parallel to the x-axis the positive direction is upward.* All lines should be read in the positive direction unless it is

intended that their lengths are to be considered negative, Since coördinates are the measures of directed lines, they must be read up or to the right if positive, and down or to the left if negative.

Exercise 1. Prove that for any position of the point C on a directed line passing through the points A and B,

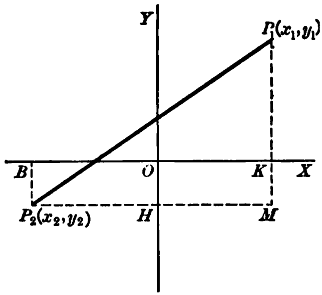
$$AB = AC + CB. \qquad \text{(Three cases.)}$$

Exercise 2. Prove that the distance between any two points having the same ordinate is the difference of their abscissas, or

$$d = x_1 - x_2. \qquad \text{(Three cases.)}$$

6. Distance Formula. — *The distance between two points, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, whose coördinates are known is given by the formula,*

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{1}$$



For $\qquad d = P_2P_1 = \sqrt{\overline{P_2M}^2 + \overline{MP_1}^2}.$

But $\qquad P_2M = P_2H + HM = OK - OB = x_1 - x_2,$

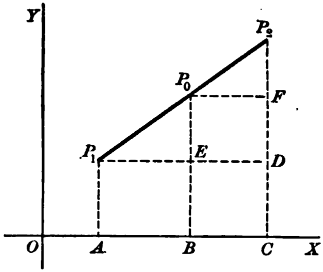
and $\qquad MP_1 = MK + KP_1 = KP_1 - BP_2 = y_1 - y_2.$

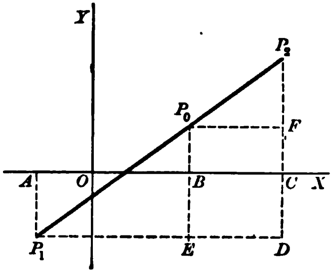
Substituting, we have the formula.

NOTE. — In this demonstration and those which follow, the fact that the formulas are true for all positions of the points and lines involved is of fundamental importance. The student should draw the figure for several possible positions and satisfy himself that the demonstration covers all cases.

7. Point of Division Formulas. — *The coördinates* (x_0, y_0) *of a point* P_0 *dividing a line joining* $P_1(x_1, y_1)$ *and* $P_2(x_2, y_2)$ *in a given ratio* $r_1 : r_2$ *are given by the formulas*

$$x_0 = \frac{r_2 x_1 + r_1 x_2}{r_1 + r_2}, \qquad y_0 = \frac{r_2 y_1 + r_1 y_2}{r_1 + r_2}. \tag{2}$$





In either figure, we have, by similar triangles,

$$P_1P_0 : P_0P_2 = P_1E : P_0F.$$

But $P_1P_0 : P_0P_2 = r_1 : r_2$ by hypothesis. Also $P_1E = OB - OA = x_0 - x_1$, and $P_0F = OC - OB = x_2 - x_0$.

Substituting, $$\frac{r_1}{r_2} = \frac{x_0 - x_1}{x_2 - x_0}.$$

Solving for x_0, $$x_0 = \frac{r_1 x_2 + r_2 x_1}{r_1 + r_2}.$$

We derive the formula for y_0 from the triangles P_0FP_2 and P_1EP_0 in the same manner.

When P_0 is the mid-point of the line, $r_1 = r_2$ and these formulas become:

$$x_0 = \tfrac{1}{2}(x_1 + x_2), \qquad y_0 = \tfrac{1}{2}(y_1 + y_2), \tag{2 a}$$

which are called the *mid-point formulas.*

Exercise 3. Derive the distance formula when

(*a*) P_1 lies in the fourth quadrant and P_2 in the second;
(*b*) P_1 lies in the second quadrant and P_2 in the third;
(*c*) P_1 lies in the third quadrant and P_2 in the fourth.

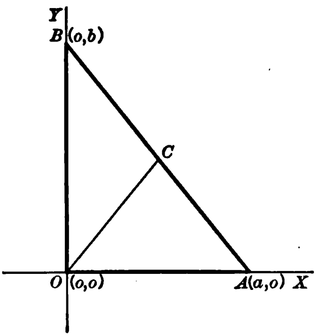
Exercise 4. State the rule expressed by (a) the distance formula, (b) the mid-point formula.

Exercise 5. Derive the mid-point formulas directly from a figure.

Exercise 6. Derive the point of division formulas for P_0 on P_1P_2 produced.

Hint. — Note the sign of $r_1 : r_2$.

8. Geometric Applications. — By means of these formulas and others which will follow, many of the theorems of plane geometry can be proved in a very simple manner.



Example. — Show by analytic means that the mid-point of the hypothenuse of any right triangle is equidistant from the vertices.

Solution. — Let the lengths of the legs be a and b. Place the right angle at the intersection of the coördinate axes. Then the vertices are (0, 0), (a, 0), (0, b). Let C be the mid-point of the hypothenuse. By formula 2 a its coördinates are $\left(\frac{a}{2}, \frac{b}{2}\right)$. The distance formula gives

$$OC = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \tfrac{1}{2}\sqrt{a^2 + b^2}.$$

The same result is obtained for CB and AC, which proves the theorem.

In this problem the lengths of the sides were expressed by letters so as to insure that OBA should represent *any* right triangle. The justification for placing the right angle at the origin lies in the geometric axiom that figures may be moved about in space without altering their size and shape.

PROBLEMS

1. Find the length of the line joining (1, 3) and (– 2, 6).

Solution. — Call (1, 3) P_1 and (– 2, 6) P_2. Then $x_1 = 1$, $y_1 = 3$, and $x_2 = -2$, $y_2 = 6$. Substituting in the distance formula,

$$d = \sqrt{(1+2)^2 + (3-6)^2} = \sqrt{18} = 3\sqrt{2}.$$

2. Find the lengths of the sides of the triangles whose vertices are

(*a*) (4, 3), (2, 2), (– 3, 5);
(*b*) (– 3, 1), (7, –2), (– 6, 5);
(*c*) (5, 4), (– 3, 2), (– 3, – 6);
(*d*) (4, 5), (2, – 3), (– 6, – 3).

Ans. (*a*) $\sqrt{5}$, $\sqrt{34}$, $\sqrt{53}$.

3. Show that the following are the vertices of isosceles triangles:

(*a*) (2, 4), (5, 1), (6, 5);
(*b*) (2, 6), (6, 2), (– 3, – 3).

4. Show that (3, 3), (– 3, – 3), $(3\sqrt{3}, -3\sqrt{3})$ are the vertices of an equilateral triangle, and find the length of its medians.

5. Show that (3, 1), (6, 5), (–1, 4) are the vertices of a right triangle. What is its area and the distance between the mid-points of its legs?

6. Show that (15, 8), (23, – 7), (8, – 15), (0, 0) are the vertices of a square. Find the lengths of its diagonals.

7. Show that the points (3, 4), $(2, 1 + \sqrt{12})$, (4, 3) lie on a circle with its center at (1, 1).

8. Find the coördinates of the point dividing in the ratio 3 : 5 the segment whose extremities are:

(*a*) (5, 6) and (13, – 10);
(*b*) (– 1, 5) and (3, – 9);
(*c*) (– 6, 3) and (8, – 7). *Ans.* (*a*) (8, 0).

9. In each of the following find the coördinates of the point three fourths of the distance from the first to the second point:

(*a*) (5, 6) and (13, – 8);
(*b*) (– 1, 5) and (3, – 4);
(*c*) (– 6, 8) and (3, – 2). *Ans.* (*a*)(11, – 4.5).

Hint. — First determine the ratio of the segments.

10. Find the point of intersection of the medians of the triangles whose vertices are the points in Problem 2. *Ans.* (*a*) $(1, \frac{10}{3})$.

Hint. — We know from plane geometry that it is two thirds of the distance from a vertex to the middle of the opposite side.

11. Find a point 10 units distant from the point (– 3, 6) and with the abscissa 3.

12. Find a point equidistant from the points (13, 8), (6, 15) and (– 4, – 9).

13. Two of the vertices of an equilateral triangle are (1, 4) and (3, – 2). Find the other vertex.

14. Prove analytically that:

(*a*) the diagonals of a square are equal;
(*b*) the diagonals of a rectangle are equal;
(*c*) a line joining the middle points of two sides of a triangle is one half the third side;
(*d*) the median of a trapezoid is one half the sum of the bases.

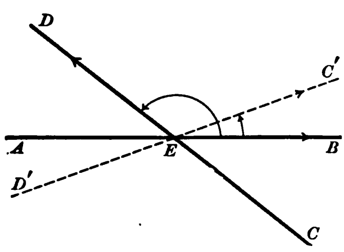
15. Prove analytically that in any triangle:

(*a*) the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides decreased by twice the product of one of those sides and the projection of the other upon it;
(*b*) the sum of the squares of two sides is equal to twice the square of one half the third side, increased by twice the square of the median on that side;
(*c*) the sum of the squares of the medians is equal to three fourths the sum of the squares of the three sides.

16. Prove analytically that:

(*a*) the sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals;
(*b*) the sum of the squares of the four sides of any quadrilateral is equal to the sum of the squares of the diagonals increased by four times the square of the line joining the mid-points of the diagonals.

9. Angle between two Lines. — *The angle of intersection of two directed lines is understood to be the angle between their positive directions; i.e.* the angle of intersection of AB and CD is BED. Therefore in Cartesian geometry the value of any angle lies between 0 and π. For example, suppose the line CD to be rotated around E so that the angle BED increases from 0 until ED reaches and passes the position EA. After that the positive direction on CD becomes EC' and the angle between the lines is BEC'.



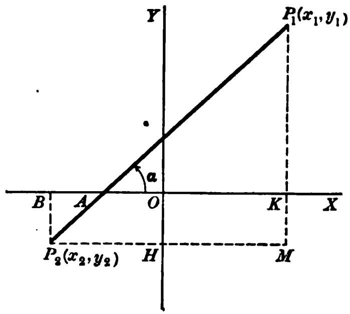
10. Inclination and Slope. — *The inclination of a line is its angle of intersection with the x-axis.* If it is parallel to the x-axis, its inclination is 0.

The slope of a line is the tangent of its inclination. We denote the inclination by the letter α, the slope by m. Thus $m = \tan\alpha$. When m is positive, α is acute; when m is negative, α is obtuse, and conversely.

The slope fixes the direction of a line. Thus a straight line is determined when its slope and one point on it are known.

11. The Slope Formula. — *The slope of a line passing through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the formula*

$$m = \frac{y_1 - y_2}{x_1 - x_2} \qquad (3)$$



Take the case where P_1 lies in the first quadrant and P_2 in the third.

Then $$\tan\alpha = \frac{KP_1}{AK} = \frac{MP_1}{P_2M}.$$

Now $$MP_1 = MK + KP_1 = KP_1 - BP_2 = y_1 - y_2,$$

and $$P_2M = P_2H + HM = OK - OB = x_1 - x_2.$$

Substituting, we have the formula.

Exercise 7. Derive the slope formula when

(*a*) P_1 is in the fourth quadrant and P_2 in the second;

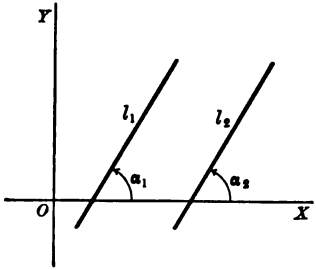
(*b*) P_1 is in the second quadrant and P_2 in the third;

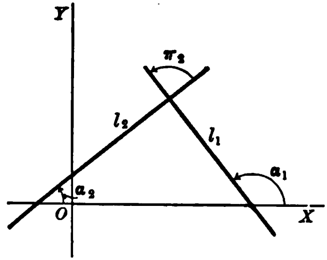
(*c*) P_1 is in the third quadrant and P_2 in the fourth;

(*d*) both points are in the first quadrant and the inclination is greater than 90°.

Exercise 8. State the rule expressed by the slope formula.

12. Parallelism and Perpendicularity. — *If two lines are parallel their slopes are equal, and conversely; if they are perpendicular their slopes are negative reciprocals, and conversely.*





If l_1 and l_2 are parallel, obviously $\alpha_1 = \alpha_2$, whence the slopes are equal.

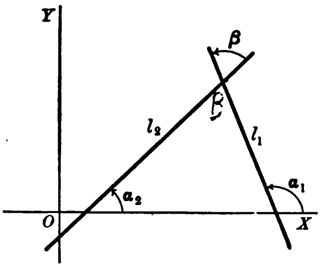
If l_1 is perpendicular to l_2, we have $\alpha_1 = \alpha_2 + \frac{\pi}{2}$, whence $\tan \alpha_1 = -\cot \alpha_2 = -\frac{1}{\tan \alpha_2}$. This gives $m_1 = -\frac{1}{m_2}$, or $m_1 m_2 = -1$.

Let the student prove the converse theorems.

Summarizing, we have:

Condition for parallelism, $\quad m_1 = m_2$; $\quad$ (4)

Condition for perpendicularity, $\quad m_1 m_2 = -1$. $\quad$ (5)



13. The Angle Formula. — *The angle between two lines is given by the formula*

$$\tan \beta = \frac{m_1 - m_2}{1 + m_1 m_2}, \qquad (6)$$

m_1 denoting the slope of the line of greater inclination.

Separate into 2 limits

Obviously $\alpha_1 = \alpha_2 + \beta$ and $\beta = (\alpha_1 - \alpha_2)$, $a_1 = a_2 + \beta$

$\therefore \beta = a_1 - a_2$

whence
$$\tan\beta = \frac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2}$$
$$= \frac{m_1 - m_2}{1 + m_1 m_2}.$$

$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$

Exercise 9. Derive the angle formula for the case when the point of intersection is below the x-axis.

PROBLEMS

1. Find the slope of the line joining

(*a*) (2, 3), (3, 5);
(*b*) (3, 8), (− 6, − 6);
(*c*) ($\sqrt{2}$, $\sqrt{3}$), (− $\sqrt{3}$, $\sqrt{2}$);
(*d*) ($a + b$, c), (a, $b + c$).

2. What is the inclination of the line joining

(*a*) (3, 3) and (− 3, − 3);
(*b*) (− 3, 2) and (− 4, 3);
(*c*) (5, 0) and (6, $\sqrt{3}$);
(*d*) (− 3, 0) and (− 2, $\sqrt{3}$);
(*e*) (0, − 4) and (− 3, 1);
(*f*) (0, 0) and (− $\sqrt{3}$, − 5)?

Ans. (*a*) 45°.

3. Prove by means of slopes that (2, 3), (6, − 3) and (− 2, 9) are on the same straight line.

4. Prove that (4, 7) is on the line joining the points (6, 6) and (2, 8), and is equidistant from them.

5. Prove that the line joining (6, − 3) and (2, 8) is perpendicular to the one joining (0, 0) and (11, 4).

6. What is the inclination of a line parallel to $Y'Y$? perpendicular to $Y'Y$? What are the slopes of these lines?

7. Find the angle between

(i) lines (*a*) and (*d*) in Problem 2;
(ii) lines (*b*) and (*e*) in Problem 2;
(iii) lines (*c*) and (*f*) in Problem 2.

Ans. (i) 15°.

8. Find the angle between the line from $(4, -2)$ to $(-3, 6)$ and the line bisecting the first quadrant.

9. Prove by means of slopes that the following points are the vertices of a right triangle and find its angles:

(*a*) $(5, 0), (8, 4), (1, 3)$;
(*b*) $(5, 5), (2, 8), (10, 10)$.

10. Prove by means of slopes that the following points are the vertices of a parallelogram and find its angles:

(*a*) $(-1, -2), (0, 1), (3, 4), (2, 1)$;
(*b*) $(3, -3), (4, 0), (7, 3), (6, 0)$.

11. Prove that $(0, -2)$, $(4, 2)$, $(0, 6)$, and $(-4, 2)$ are the vertices of a square.

12. Find the slopes of the sides and the angles of the triangle whose vertices are:

(*a*) $(-2, 2), (4, 2), (1, 5)$; (*b*) $(1, 1), (-1, -1), (5, -5)$.

13. Prove analytically that

(*a*) the diagonals of any square are perpendicular;
(*b*) the median of a trapezoid is parallel to the bases;
(*c*) the lines joining the mid-points of the sides of any quadrilateral form a parallelogram;
(*d*) the lines joining the mid-points of the sides of a rectangle form a rhombus.

14. What is the relation between two lines for which $m_1 = -m_2$?

15. Three vertices of a parallelogram are $(1, 6)$, $(2, 5)$, and $(4, -2)$. Find the fourth vertex.

16. Find a point on the line joining $(2, 3)$ and $(-4, 6)$ and 2 units distant from the latter point.

17. Derive formula (5) from formula (6).

14. Equations and Graphs.—The coördinates x, y, if no restriction is placed on their values, represent any point in the plane. If, however, the values of x and y are subject to certain conditions, points having these coördinates will lie upon certain lines or curves. For example, if y is unrestricted, but x always equals -6, all such points will lie upon a line parallel to the y-axis and 6 units to the left. Again, the locus of all points whose abscissas and ordinates

are equal is the bisector of the first and third quadrants. Such restrictions upon coördinates are expressed by means of equations. Thus, the equation of the first locus is $x = -6$, of the second $x = y$. As a general definition, we have:

The equation of a locus is an equation satisfied by the coördinates of all points lying on the given locus, and conversely. The curve which contains all points whose coördinates satisfy the given equation and no other points is called the *locus* or *graph* of the equation.

The derivation of equations of loci and the study of graphs by means of their equations form the chief part of elementary Analytic Geometry.

15. Plotting of Graphs. — When a locus is given by its equation, the shape of the curve is sometimes evident from the form of the equation, as in examples mentioned above, and as we shall see in more complicated problems later. Usually the graph is constructed by a process called plotting the graph of the equation, in which we proceed as follows:

Solve the equation for y in terms of x.

Set x equal to convenient positive and negative values (generally integral) *and compute the corresponding values of y.* Each pair of values of x and y is a solution of the equation, and hence the point of which these are the coördinates lies on the curve by the definition of the locus.

Make a table of values by arranging these pairs in order according to the magnitudes of the values of x.

Plot the points thus tabulated and join them by a smooth curve in the order of the table. This gives an approximation to the true curve which becomes more exact when a larger number of points is plotted.

Remarks. — It is sometimes more convenient to solve for x in terms of y. In this case the above is applicable on interchanging x and y.

If the solution for y in terms of x involves a square root, the double sign must be used with the radical and in general two points will be found for each value of x. A similar remark applies to a solution for x in terms of y.

Sometimes the values of x or y will be so large that it is difficult to draw the curve on the plotting paper. In this case use one scale for x and another for y.

When the points plotted do not show the form of the curve, look for an error in the table of values. If there is no error, assume intermediate fractional values of x or y and plot the corresponding points.

The table of values should be extensive enough to show the form of the curve completely. If it extends to infinity, the plot should contain all parts with considerable curvature and should extend far enough to show the direction of the curve beyond the limits of the paper.

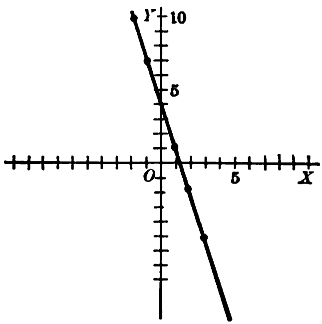
EXAMPLE 1. — Plot the graph of $3x + y - 4 = 0$.

x	y
-2	10
-1	7
0	4
1	1
2	-2
3	-5
etc.	

Solving for y,

$$y = -3x + 4.$$

Substituting $x = -2$, -1, 0, 1, 2, 3, we get $y = 10, 7, 4, 1, -2, -5$ respectively. Thus points on the line are $(-2, 10)$, $(-1, 7)$, $(0, 4)$, etc. The table of values is at the left and the plot at the right.



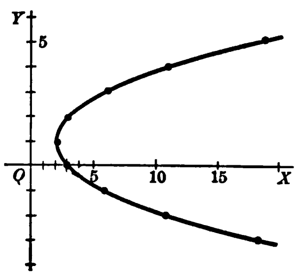
EXAMPLE 2. — Plot the graph of $y^2 - 2y = x - 3$.

y	x
-3	18
-2	11
-1	6
0	3
1	2
2	3
3	6
4	11
5	18

Here it is easier to solve for x in terms of y:

$$x = y^2 - 2y + 3.$$

In the plot each unit on the y-axis has been taken equal to two units on the x-axis. This is because the values of x are so much larger than those of y that if the same scale were used on both axes, the curve would be hard to draw. In such cases the scale used should be indicated on the graph.

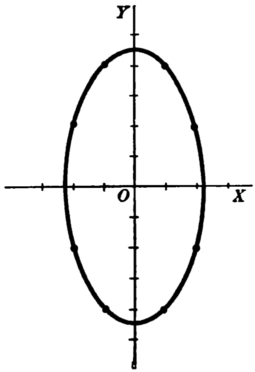


EXAMPLE 3. — Plot the graph of $4x^2 + y^2 = 20$.

x	y
-3	imag.
-2	± 2
-1	± 4
0	$\pm 2\sqrt{5}$ $= \pm 4.5$
1	± 4
2	± 2
3	imag.

Solving for y in terms of x,

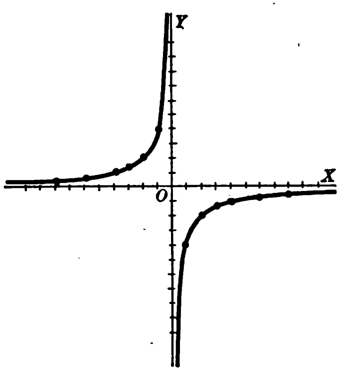
$$y = \pm\sqrt{20 - 4x^2}$$
$$= \pm 2\sqrt{5 - x^2}.$$



Here no part of the curve can lie outside of $x = +3$ and $x = -3$, whereas there are two points corresponding to $x = \pm 2$. To find out where the curve crosses the x-axis, we set $y = 0$ in the equation and solve, getting $x = \pm\sqrt{5} = \pm 2.2+$. Adding the points $(2.2+, 0)$ and $(-2.2+, 0)$ to the plot, we are able to draw the entire curve.

EXAMPLE 4. — Plot the graph of $xy = -4$, or $y = -\dfrac{4}{x}$.

x	y	x	y
1	-4	-1	4
2	-2	-2	2
3	$-\frac{4}{3}$	-3	$\frac{4}{3}$
4	-1	-4	1
6	$-\frac{2}{3}$	-6	$\frac{2}{3}$
8	$-\frac{1}{2}$	-8	$\frac{1}{2}$



Since x cannot equal 0*, we must take fractional values of x between ± 1 in order to find the shape of the curve. These give additional points:

x	y	x	y
$\frac{1}{2}$	-8	$-\frac{1}{2}$	8
$\frac{1}{4}$	-16	$-\frac{1}{4}$	16

Adding these to the plot, we see that the curve goes to infinity along the x- and y-axes in the second and fourth quadrants.

* This is due to the fact that the substitution of $x = 0$ in the equation involves division by 0, an impossible operation. Since y becomes indefinitely great as x *approaches* 0, it is sometimes stated that for $x = 0$, $y = \infty$, but this abbreviation should not be allowed to obscure the fact that for $x = 0$ there is *no* point on the curve.

PROBLEMS

Plot the graphs of the following equations :

1. $x = 6$.
2. $y = 0$.
3. $y = -2$.
4. $3y = x$.
5. $y = -2x$.
6. $3x + 4y = 0$.
7. $4x - 3y = 6$.
8. $x + y = 1$.
9. $y^2 = x^2$.
10. $y = x^2 - 5x + 4$.
11. $x^2 - 3x + 2y = 6$.
12. $x = y^2 - 6y + 5$.
13. $y^2 - y = x + 2$.
14. $x^2 + y^2 = 16$.
15. $x^2 + y^2 = 25$.
16. $x^2 + 4y^2 = 4$.
17. $9x^2 + y^2 = 36$.
18. $x^2 + 2y^2 = 2$.
19. $25x^2 + 16y^2 = 400$.
20. $x^2 - y^2 = 9$.
21. $y^2 = 25 + x^2$.
22. $x^2 + 4x + 4 = y^2 - 3y$.
23. $x^2 - 9y^2 = 36$.
24. $9x^2 - y^2 + 9 = 0$.
25. $xy = 12$.
26. $2xy = -15$.
27. $y = x^3 - 9x^2$.
28. $y^3 = x^2$.
29. $xy = y + 2$.
30. $xy = 3x - 1$.
31. $xy - y^2 = 10$. (Solve for x.)
32. $x + y - 2xy = 0$.
33. Find two of the above curves which pass through
    (*a*) $(0, 0)$; (*b*) $(-1, +1)$; (*c*) $(1, 1)$. ✓

16. Derivation of Equations. — The derivation of equations of loci or curves is a process depending largely upon the ingenuity of the solver of the problem. However, the following general suggestions will be helpful.

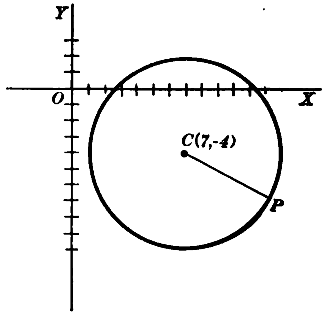
1. *Take the origin and axes in a convenient position.* The origin will usually be a fixed point mentioned in the problem, or the point midway between two such points, with one of the axes passing through them. This does not involve any loss of generality as far as the *locus* is concerned, since the locus is independent of its position in the plane; while a correct choice of axes will greatly simplify the form of the equation and the work of deriving it.

2. *Mark $P(x, y)$ as any point satisfying the given conditions and therefore on the curve.* Draw its coördinates if necessary.

3. *Draw any line suggested by the data of the problem.*

4. *Express the conditions of the problem in an equation containing x, y, and the given constants.* For this purpose it will be necessary to find some formula or principle of geometry relating to the coördinates and given constants.

5. *Simplify the equation.* It should contain all the given numerical or arbitrary constants, but *no variables other than x and y.*



EXAMPLE 1. — Find and plot the locus of points distant 6 units from the point $(7, -4)$.

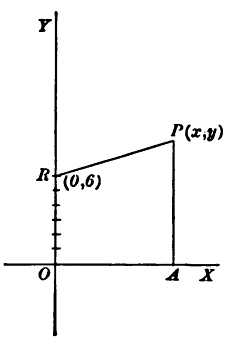
Solution. — Here the coördinate axes are fixed by the statement of the problem. Take any point $P(x, y)$ as one satisfying the conditions of the problem and draw CP. Since CP has the constant length 6, the distance formula is suggested. This gives at once

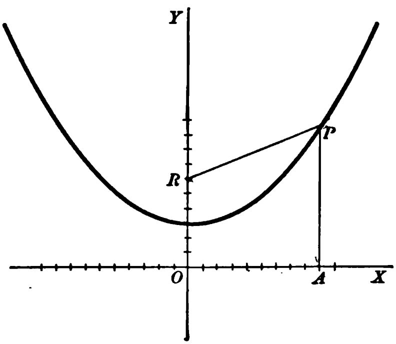
$$6 = \sqrt{(x-7)^2 + (y+4)^2}, \text{ or } (x-7)^2 + (y+4)^2 = 36.$$

We can either plot the curve by points, or, since we know that it is a circle, draw it with compasses.

EXAMPLE 2. — A rock in the ocean lies 6 miles off a stretch of straight coast and a ship moves so as to be always equidistant from the rock and the coast. Find the locus of the ship.

Solution. — Take the coast as the x-axis and let the y-axis pass through the rock, which will have the coördinates $(0, 6)$. Take a point $P(x, y)$ and draw its ordinate AP. Then $OA = x$, $AP = y$. P is to be equidistant





from R and OA. This suggests drawing RP, AP being already drawn. By the conditions of the problem $RP = AP$. To get the length of RP use the distance formula, which gives

$$RP = \sqrt{x^2 + (y-6)^2}. \quad \text{But its equal } AP = y.$$

$$\therefore \sqrt{x^2 + (y-6)^2} = y, \text{ whence } x^2 - 12y + 36 = 0.$$

The derivation of the above equations shows that the coördinates of all points which lie on the locus satisfy the equation obtained. To show that the equation fulfills the second requirement of the definition of the locus, namely, that any point the coördinates of which satisfy the equation, lies on the locus, it is merely necessary to see if the various steps taken in the derivation of the equation can be retraced. To illustrate, in Example 2 any point $P(x, y)$, whose coördinates satisfy the equation $x^2 - 12y + 36 = 0$, must also satisfy the previous equation since it can be reduced to that form by reversing the steps by which the latter was obtained. But the relation $\sqrt{x^2 + (y-6)^2} = y$ simply says that the point P is equidistant from the x-axis and R, which was the condition stated in the problem.

PROBLEMS

1. What is the equation of a straight line

(*a*) parallel to the x-axis and 6 units above it;
(*b*) parallel to the y-axis and 6 units at the left?

2. Find and plot the equation of the locus of points

(*a*) three times, (*b*) m times

as far from the x-axis as from the y-axis.

3. Solve Example 2, page 25, when

(*a*) the origin is at R;
(*b*) the origin is midway between O and R.

4. Find and plot the equation of the locus of points equidistant from

(*a*) (3, 2) and (6, − 2);
(*b*) (− 4, 3) and (6, 0);
(*c*) (− 5, 0) and (0, 4);
(*d*) (0, 0) and (*a*, 0).

5. Find and plot the equation of the circle having

(*a*) radius 3 and center $(-5, 6)$;
(*b*) radius 6 and center $(1, -3)$;
(*c*) radius 2 and center $(0, 0)$;
(*d*) radius r and center (h, k).

6. Find and plot the equation of the locus of points equidistant from

(*a*) the line $y = 6$ and the point $(3, 0)$;
(*b*) the line $x = 4$ and the point $(-2, 0)$;
(*c*) the line $x = -8$ and the point $(0, -5)$;
(*d*) the line $x = 0$ and the point $(p, 0)$.

7. Find the locus in Problem 6 when the distance from the point is twice that from the line.

8. Find the locus in Problem 6 when the distance from the line is twice that from the point.

9. Find the equation of the straight line passing through

(*a*) $(3, 4)$ and $(5, -2)$; (*c*) $(1, 1)$ and $(-2, -2)$;
(*b*) $(4, 3)$ and $(-1, 6)$; (*d*) $(a, 0)$ and $(0, b)$.

10. Find the equation of the straight line passing through $(-2, 3)$ and of inclination

(*a*) $120°$; (*b*) $\frac{3\pi}{4}$; (*c*) $150°$; (*d*) $\frac{\pi}{6}$.

11. The distance between two fixed points is $2c$. Find the locus of a point moving so that the sum of the squares of its distances from the points is $4c^2$.

12. The ends of a straight line of variable length rest on two perpendicular lines. Find the locus of the middle point if the area of the triangle formed is constant.

13. The base of a triangle is of length $2a$. Find the locus of the vertex if the vertical angle is $90°$.

17. Functional Variables. — The symbols used in mathematics represent two kinds of quantities, variables and constants.

A *variable* is a quantity which takes on an unlimited number of values; *e.g.* the velocity of a falling body, the abscissa of a point moving along a curve, etc., are variable quantities.

A *constant* is a quantity having a fixed value. There are two kinds of constants: *absolute* constants, which have the

same value in all problems, as 2, -6, π; and *arbitrary* constants, which may have any value assigned, but keep the same value in a given discussion, as the quantities m_1 and m_2 in the discussion of parallel and perpendicular lines in § 12.

When two variables are connected by some law such that the value of one depends upon that of the other, the first variable is said to be a function of the other. The first variable is called the *dependent* and the second the *independent* variable. Thus the area of a circle is a function of the radius, the pressure of steam is a function of the temperature, etc. Such relations are commonly expressed by means of equations, as, in the case of the formula expressing the area of a circle, $A = \pi r^2$.

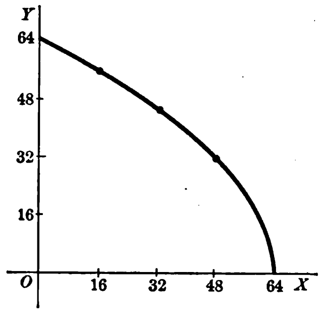
18. Functional Notation. — The expression $f(x)$ is used to denote any function of x; it is read "function of x," or "f of x." Similarly $f(x, y)$ stands for a function of both x and y. To denote a different function of x, some other letter is used, as $F(x)$ or $g(x)$. Since functional relations are usually expressed by means of equations, the expression $f(x)$ is ordinarily an abbreviation for some combination of terms containing x and no other variable.

If an equation in x and y is solved for y, we regard x as the independent variable and y as a function of x. Thus, if x and y are connected by the relation $x^2 + 4y^2 = 4$, $y = \pm \frac{1}{2}\sqrt{4 - x^2}$ may be denoted by the abbreviation $y = f(x)$, where $f(x)$ stands for the quantity $\pm \frac{1}{2}\sqrt{4 - x^2}$. The symbols $f(1)$, $f(-2)$ represent the values of the function when 1 and -2 respectively are substituted for x. For this function

$$f(1) = \pm \tfrac{1}{2}\sqrt{4-1} = \pm \tfrac{1}{2}\sqrt{3},\ f(-2) = \pm \tfrac{1}{2}\sqrt{4-4} = 0.$$

If it is not desired to solve the equation for either variable, we may transpose all terms to the first member and denote this form of the equation by the expression $f(x, y) = 0$.

19. Graphs of Functions. — If we plot the graph of an equation expressing a functional relation, the graph may be regarded as a pictorial representation of the relation between the variables. Consider, for example, the relation between the velocity of a ball thrown vertically up with an initial velocity of 64 feet per second and the height which it has attained at any time. By a physical formula



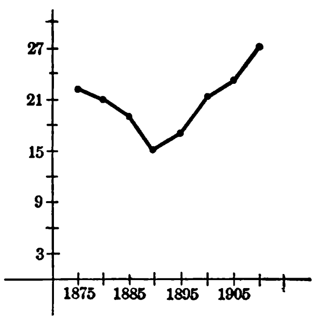
$$v^2 = 64^2 - 64\,h.$$

h	v
0	64
16	55
32	45
48	32
64	0

Let each unit on the x-axis represent 1 foot and each unit on the y-axis represent a velocity of 1 ft. per second. For each value of h the corresponding ordinate represents the velocity of the ball at that height.

In presenting statistical information the graph is constantly used. Here, as a general thing, the functional relations cannot be expressed by means of equations, and the graphs are plotted from observed data. The following table gives the debt of the United States for the years 1875 to 1910, the unit being \$100,000,000, and the adjoining graph presents the same information.

Year	Debt in Hundreds of Millions
1875	22
1880	21
1885	19
1890	15
1895	17
1900	21
1905	23
1910	27



PROBLEMS

1. Write the equation $2x^2 + y^2 = 4xy$ in the three forms $y = f(x)$, $x = f(y)$, and $f(x, y) = 0$.

Solution. — To express y in terms of x, solve the equation for y,

$$y = \frac{4x \pm \sqrt{16x^2 - 8x^2}}{2} = 2x \pm x\sqrt{2}.$$

Hence $\quad y = f(x) = 2x \pm x\sqrt{2}.$

In like manner $\quad x = f(y) = y \pm \frac{1}{2}y\sqrt{2},$

and $\quad f(x, y) = 2x^2 - 4xy + y^2 = 0.$

2. Write each of the following equations in the three forms $y = f(x)$, $x = f(y)$, and $f(x, y) = 0$:

(*a*) $xy + y = 6$; (*c*) $x^2 + 2x = y^2 - 6y$;
(*b*) $x^2 + 9y^2 = 9$; (*d*) $y = (x + 1)^3$.

3. If $f(x) = x^3 - 2x^2 + 3x$, find the value of $f(1)$, $f(0)$, $f(-3)$, $f(-x)$.

4. If $f(x) = 6x^4 - 5x^2 + 3$, show that $f(-x) = f(x)$ identically.

5. If $f(x, y) = 8x^2 + 3y - y^2$, show that $f(-x, y) = f(x, y)$ identically. Does $f(x, -y) = f(x, y)$?

6. Write a function of x and y such that

(*a*) $f(x, -y) = f(x, y)$ identically;
(*b*) $f(-x, -y) = f(x, y)$ identically;
(*c*) $f(-x, y) = f(x, y)$ identically.

7. Express the radius of a sphere as a function of the surface and plot the graph of the function.

8. The area of a triangle of variable base and altitude is always 16 square feet. Express the base as a function of the altitude and plot the graph of this function.

9. A point moves around the circumference of a circle of radius 5. Express its distance from one end of a fixed diameter as a function of its distance from the other end and plot the graph of this function.

10. A rectangle of varying size has one side 3 inches longer than the other. Express the area in terms of the shorter side and plot the graph of this function.

11. From an almanac find the length of daylight on the first of each month during the year and draw a graph illustrating this data.

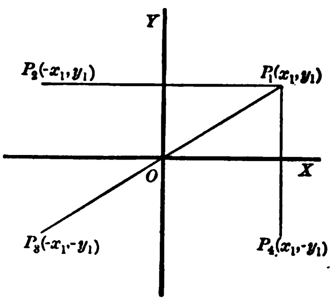
20. Discussion of Equations. — Since it is possible to plot but a few points on a curve, the graph is always more or

less inaccurate. To discover the properties of a curve it is necessary to study, or *discuss*, its equation. The purpose of this discussion is threefold: it gives *exact* information regarding the curve, it furnishes a check upon the accuracy of the plot, and it usually facilitates the labor of plotting. The properties that can be conveniently studied in most curves are the *intercepts*, *symmetry*, and *extent* of the curve.

21. Intercepts. — The intercepts of a curve are the distances from the origin to the points where it meets the axis. To find the x-intercepts, set $y = 0$ and solve the resulting equation; to find the y-intercepts, set $x = 0$ and solve.

22. Symmetry. — *Definitions. The* **axis of symmetry** *of two points is the perpendicular bisector of the line joining them. The* **center of symmetry** *of two points is the point midway between them.*

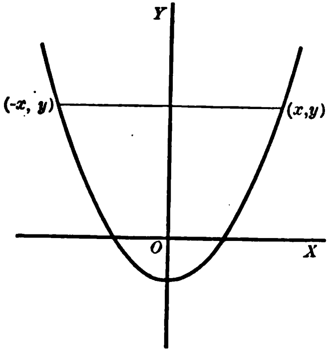
A curve is symmetrical with respect to an axis or to a center when each point of the curve has its symmetrical point on the curve.



Thus the points (3, 4) and (−3, 4) are symmetrical with respect to the y-axis; (3, 4) and (3, −4) with respect to the x-axis; and (3, 4) and (−3, −4) with respect to the origin. From the figure we can readily establish the general principle:

Two points are symmetrical with respect to the x-axis when their abscissas are the same and their ordinates differ only in sign; to the y-axis when their ordinates are the same and their abscissas differ only in sign; to the origin when their respective coördinates differ only in sign.

The proof is left to the student.



Applying this theorem to curves, we see that a curve is symmetrical with respect to the y-axis, if for each point (x, y) on the curve, the point $(-x, y)$ is also on the curve. The symmetry of a curve may be determined by inspection of the equation according to the following theorem:

If the substitution in an equation of

$-x$ **for** x	**gives an equation reducible to its original form, the curve is symmetrical with respect to the**	y**-axis;**
$-y$ **for** y		x**-axis;**
$-x$ **for** x **and** $-y$ **for** y		**origin.**

The proof of the first statement is as follows:

Let the equation be written in the form $f(x, y) = 0$. Let $P(x_1, y_1)$ be any point on the curve and $Q(-x_1, y_1)$ be its symmetrical point with respect to the y-axis.

By hypothesis, $f(-x, y) \equiv f(x, y)$; hence $f(-x_1, y_1) = f(x_1, y_1)$.

But P is on the curve, hence $f(x_1, y_1) = 0$. Therefore $f(-x_1, y_1) = 0$, and hence Q is on the curve.

Thus the definition of symmetry is satisfied and the theorem proved. The other cases are treated in exactly the same manner.

The above figure is the graph of $x^2 - 2y - 1 = 0$, an equation satisfying the test for symmetry with respect to the y-axis. Solving for x, we have $x = \pm\sqrt{2y+1}$, and the table of values illustrates various pairs of symmetrical points.

y	x
$-\frac{1}{2}$	0
0	± 1
1	$\pm\sqrt{3}$
2	$\pm\sqrt{5}$
4	± 3

A curve which is symmetrical with respect to both the x and y-axes is symmetrical with respect to the origin, but the converse is not true (*e.g.* $y = x^3$). Moreover, a curve may

have axes of symmetry other than the coördinate axes. In this case, if there are two perpendicular axes of symmetry, their intersection will be a center of symmetry.

Exercise 10. Prove that if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are symmetrical with respect to the x-axis, then $x_1 = x_2$, and $y_1 = -y_2$.

Exercise 11. Write a proof of the third test for symmetry.

23. Extent. — To investigate the extent of a curve, we solve its equation for y in terms of x, and for x in terms of y. If either operation gives rise to radicals of even degree involving one of the variables, say x, then values of x which make the expression under the radical negative must be excluded, for the corresponding values of y would be imaginary.

To illustrate, consider the equation, $4x^2 - 9y^2 = 36$.

Solving. $y = \pm \frac{2}{3}\sqrt{x^2-9}, \qquad x = \pm \frac{3}{2}\sqrt{y^2+4}.$

For $x^2 < 9$, or for values of x between ± 3, $x^2 - 9$ is negative, and therefore y is imaginary. Hence such values of x must be excluded from the table of values and the curve lies wholly without that part of the plane bounded by the lines $x = \pm 3$.

As $y^2 + 4 > 0$ for all values of y, no value of y needs to be excluded.

PROBLEMS

1. Discuss the equation $4x^2 + y^2 - 16x = 0$ and plot its graph.

Solution. — The x-intercepts are 0 and 4; the only y-intercept is 0.

x	y
0	0
1	± 3.4
2	± 4
3	± 3.4
4	0

Symmetry: With respect to the x-axis.

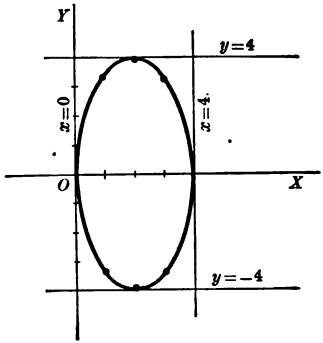
Extent:

$x = 2 \pm \frac{1}{2}\sqrt{16 - y^2}.$

Hence y^2 must be ≤ 16 and the curve lies between the lines $y = \pm 4$.

$y = \pm\sqrt{4x(4-x)}.$

If x is > 4 or < 0, the factors under the radical have unlike signs and their product is negative. Hence such values of x must be excluded and the curve lies between the lines $x = 0$ and $x = 4$.

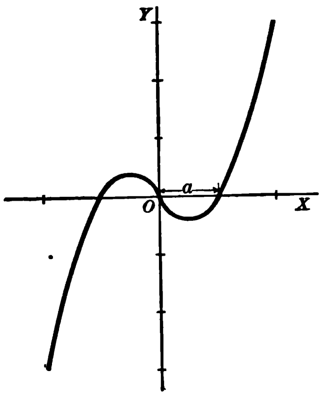


2. Discuss the equation $y = x^3 - a^2x$ and plot its graph.

Solution. — This equation contains an arbitrary constant a. It will have a different graph for each value given a, but the properties obtained by the discussion will be common to all such curves.

Intercepts: Putting $x = 0$, the y-intercept is 0. Putting $y = 0$, $x^3 - a^2x = 0$; the x-intercepts are 0, $\pm a$.

Symmetry: The substitution of $-x$ for x, or $-y$ for y, changes the equation, but if $-x$ is put for x and also $-y$ for y, we get



$$-y = (-x)^3 - a^2(-x),$$

which reduces to

$$y = x^3 - a^2x.$$

Thus the curve is symmetrical with respect to the origin.

Extent: Since this equation is of the third degree, it will have at least one real solution for any value of y. Hence we shall not solve it for x. It is already solved for y and no radicals appear, hence no values of x need to be excluded, for none make y imaginary. Thus x may take on any value. If the equation is put in the form $y = x(x^2 - a^2)$, it becomes evident that when $x > a$, y is positive, and when x approaches infinity, y approaches infinity. Thus the curve extends to infinity in the first quadrant. Owing to the symmetry, the path where $x < 0$ is similar, extending to infinity in the third quadrant.

x	y
0	0
$\frac{1}{2}$	$-\frac{3}{8}$
1	0
2	6
3	24

We could now sketch the curve *roughly*. A table of values for x positive gives a more accurate graph. In making the table we assign a any convenient value, as 1.

etc.

Discuss each of the following equations and plot their respective graphs:

3. $4x^2 - y = 0$.
4. $y^2 - 8x + 16 = 0$.
5. $x^2 + y^2 - 16x = 0$.
6. $x^2 + 4y^2 + 4y = 0$.
7. $x^2 + 2y^2 = 16$.
8. $9y^2 - x^2 - 36 = 0$.
9. $x^2 - y^2 + 12y = 0$.
10. $x^2 - 4y^2 + 4x = 0$.
11. $xy - 9 = 0$.
12. $2xy + 15 = 0$.
13. $y^2 = 8x^3$.
14. $y = x^3 - 9x$.

15. $x^2 = 9y^3$.

16. $y + (x-2)^3 = 0$.

17. $xy^2 = 36$.

18. $x(y^2+1) - 12 = 0$.

19. $x^2y = 1$.

20. $4x^2 + 3y^2 = 0$.

21. $y^2 = ax^3$.

22. $y^2 = 2px$.

23. $x^2 + y^2 = a^2$.

24. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

25. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

26. $x^2 = 2py$.

27. $y = ax^3$.

28. $xy = 2a^2$.

24. Symmetrical Transformations. — In case the tests for symmetry of § 22 are not satisfied, the substitutions used will transform the given equation into a new equation.

For example, if we substitute $-x$ for x in the equation of a curve symmetrical with respect to the y-axis, the equation is unchanged; but if the curve is not symmetrical with respect to the y-axis, a new equation is obtained. The question at once arises: what relation has the locus of the new equation to that of the given equation?

In this case it can be shown as in the proof of § 22, that for any point $P(x_1, y_1)$ on the locus of the given equation $f(x, y) = 0$, the symmetrical point $Q(-x_1, y_1)$ lies on the locus of the new equation $f(-x, y) = 0$. For this reason the curves are said to be symmetrical to each other with respect to the y-axis, and the transformation is called a *symmetrical transformation.* Similar reasoning applies to the other substitutions.

Summarizing, we have the following definition and theorem:

Two curves are symmetrical to each other with respect to an axis or to a center, when each point of one curve has its symmetrical point on the other.

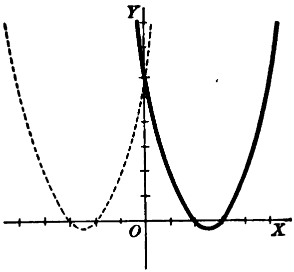
If in an equation substitution is made of

$$\left.\begin{array}{r} -x \text{ for } x \\ -y \text{ for } y \\ -x \text{ for } x \text{ and } -y \text{ for } y \end{array}\right\} \begin{array}{l} \text{the locus of the new equation} \\ \text{is symmetrical to that of the} \\ \text{old with respect to the} \end{array} \left\{\begin{array}{l} y\text{-axis;} \\ x\text{-axis;} \\ \text{origin.} \end{array}\right.$$

EXAMPLE — Write the equation of the curve symmetrical to $y = x^2 - 5x + 6$ with respect to the y-axis, and plot both curves on the same axes.

Putting $-x$ for x, we get $y = x^2 + 5x + 6$. The continuous line is the graph of the original equation, the dotted line that of the new one.

Note that the table of values for $y = x^2 + 5x + 6$ may be obtained from that for $y = x^2 - 5x + 6$ by merely changing the signs of the values of x.



TABLES OF VALUES

$y = x^2 - 5x + 6$

x	y
0	6
1	2
2	0
3	0
4	2
5	6

$y = x^2 + 5x + 6$

x	y
0	6
-1	2
-2	0
-3	0
-4	2
-5	6

25. Transformations of Equations not Altering Loci. — The following transformations of the equation leave the form of the locus unchanged.

(a) Rearrangement of terms, solving for one variable, multiplying by a constant.

These do not alter the relation between x and y.

(b) The symmetrical substitutions.

(c) Interchange of variables.

The effect of this is to interchange the axes.

PROBLEMS

1. Write the equation of the curve symmetrical to the given curve with respect to the x-axis, and plot both curves on the same axes.

(a) $y = 3x^2 - 4$; (c) $y = x^3 + 4x$;
(b) $x^2 + 4y^2 - 4y = 0$; (d) $y = x^3 - 2x^2$.

2. Write the equation of the curve symmetrical to the given curve with respect to the y-axis and plot both curves on the same axes.

(a) $y^2 = 3x - 4$; (c) $y = x^3 + 4x$;
(b) $x^2 + 4y^2 - 4x = 0$; (d) $y = x^3 - 2x^2$.

3. Write the equation of the curve symmetrical to the given curve with respect to the origin and plot both curves on the same axes.

(a) $y = 3x^2 - 2x + 4$; (c) $y = x^3 - 4x^2$;
(b) $x = 2y^2 - 6y + 8$; (d) $y = 5x - 3$.

4. Plot on the same axes the graphs of

(a) $y^2 = 3x + 5$ and $x^2 = 3y + 5$;
(b) $y = (x-1)^3$ and $x = (y-1)^3$;
(c) $x^2 - 4y^2 = 4$ and $y^2 - 4x^2 = 4$;
(d) $y^2 = x^3$ and $x^2 = y^3$;
(e) $y = x^3$ and $y = \sqrt[3]{x}$.

26. Equations whose Graphs cannot be Plotted. — Since real numbers alone can be used as the coördinates of a point, an equation which is satisfied only by imaginary values of the variables does not define a curve and has no locus. Such equations may be most easily distinguished by completing squares. If then every term is positive, the equation is obviously satisfied only by imaginary values of x and y.

PROBLEMS

1. Show that $x^2 - 2x + 2y^2 + 8y + 14 = 0$ has no locus.

Solution. — Completing the squares in x and y, we have

$$(x-1)^2 + 2(y+2)^2 + 5 = 0.$$

For any real values of x and y, $(x-1)^2$ and $(y+2)^2$ are positive; hence we have the sum of three positive numbers equal to zero, which is impossible.

2. Examine the following equations for the existence of a locus:

(a) $x^2 + 6x + y^2 - 4y + 17 = 0$;
(b) $x^2 - 3x + y^2 + 4y = -7$;
(c) $(x+2)^2 + y^2 - 6y + 8 = 0$;
(d) $x^2 + 16 = 0$;
(e) $x^4 + 4y^2 + 4 = 0$;
(f) $x^2 + 4xy + 4y^2 + 16 = 0$.

3. For what values of k has the equation $x^2 + y^2 + k = 0$ no locus?

4. What is the locus of

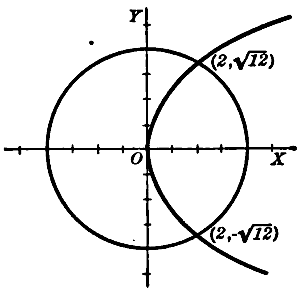
(a) $x^2 + y^2 = 0$;
(b) $(x-3)^2 + (y+3)^2 = 0$?

5. For what values of k will the locus be imaginary, a point, or a curve in each of the following:

(a) $x^2 + 9y^2 = k^2 - 4$;
(b) $x^2 - 4x + y^2 + 8y + k = 0$;
(c) $x^2 + 4x + y^2 - 6y - k = 0$;
(d) $x^2 - 9y^2 + k = 0$.

27. Intersection of Curves. — It follows from the definition of the locus of an equation that a point lies on two curves if and only if its coördinates satisfy the equation of each. Hence we conclude that the coördinates of points of intersection may be obtained by solving the equations simultaneously. If there are no real solutions, the curves do not intersect.

PROBLEMS



1. Find the points of intersection of the curves $x^2 + y^2 = 16$ and $y^2 = 6x$.

Solution. — Solving simultaneously, $x = 2$ or -8. When $x = 2$, $y = \pm\sqrt{12}$; when $x = -8$, $y = \pm\sqrt{-48}$. Thus there are but two pairs of real solutions, giving the two points $(2, \sqrt{12})$ and $(2, -\sqrt{12})$. The figure shows the curves and their points of intersection.

Find the points common to the loci of the following equations and plot the loci.

NOTE. — In many cases the points of intersection and the intercepts, together with considerations of symmetry, will be sufficient for plotting the curves without calculating tables of values.

2. $4x - y + 2 = 0$,
$x + 8y + 16 = 0$.

3. $x - y = 0$,
$y^2 + 6x = 0$.

4. $2x + 11y - 15 = 0$,
$7x - 5y - 12 = 0$.

5. $x + y - 8 = 0$,
$x^2 - 4y + 2 = 0$.

6. $x^2 + y^2 - 25 = 0$,
$y - 2x + 11 = 0$.

7. $x^2 + y^2 - 25 = 0$,
$x^2 - 3y - 21 = 0$.

8. $x^2 + y = 7$,
$y^2 - 81 = 0$.

9. $9x^2 + 4y^2 = 36$,
$9x^2 + 16y = 33$.

10. $x^2 - y^2 + 36 = 0,$
$y^2 - 5x - 32 = 0.$

11. $y^2 = 2ax,$
$x^2 = ay.$

12. $x^2 + 6y^2 = 20,$
$xy = 4.$

13. $x^2 + y^2 - 8x = 0,$
$y^2 - 8x - 2 = 0.$

14. $3x^2 + 4y^2 = 12,$
$x^2 - y^2 = -10.$

15. $y = x^3,$
$y = x^2 + 2x.$

16. $x^2 + y^2 - 6x = 0,$
$y^2 = x^3.$

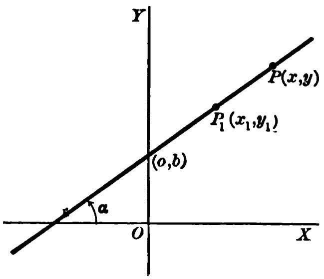
17. $xy = 4,$
$y^2 + x^2 = 9.$

CHAPTER II

THE STRAIGHT LINE

28. Equations of the Straight Line. — Several properties of the straight line, such as its determination either by two points or by a point and a direction, lead to relations between the coördinates of its tracing point that can be expressed in the form of an equation. There are several forms of the straight line equation, but for a particular line each one may in general be reduced to any of the others.

29. The Point Slope and Two-Point Forms. — Let the line be fixed by a point and a direction. Let the point be $P_1(x_1, y_1)$ and let the direction be given by the slope $m = \tan \alpha$. Let the tracing point be $P(x, y)$. By the slope formula,



1
$$m = \frac{y - y_1}{x - x_1},$$

or
$$y - y_1 = m(x - x_1). \qquad (7)$$

This is the *point slope form* of the straight line equation. It is used in writing the equation of a line when one point and the slope are known. It may also be used when two points are known, for then the slope of the line can be found at once by the slope formula (3). In the latter case it is often convenient to use the *two-point form*, which is derived as follows: Let the line be determined by the points

$P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, and let the tracing point be $P(x, y)$. The slope formula gives us at once

$$m = \frac{y - y_1}{x - x_1} \text{ and } m = \frac{y_1 - y_2}{x_1 - x_2}.$$

Equating these we have the desired form,

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}. \tag{7a}$$

30. The Slope Intercept Form. — If the slope and the y-intercept are given, the line is determined by a point on the y-axis $(0, b)$, and the slope m. Substituting in (7), we have

$$y - b = m(x - 0),$$

or

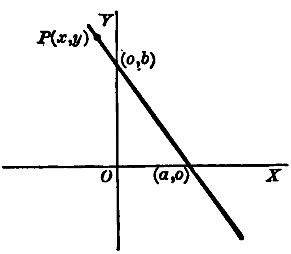
$$y = mx + b, \tag{8}$$

which is called the *slope intercept form.*

This form is used in finding the equation of a straight line when the slope and the y-intercept are known. Conversely, if an equation can be reduced to form (8), its locus is a line whose slope is the coefficient of x and whose y-intercept is the constant term in the reduced form.

31. The Intercept Form. — Suppose the intercepts of the line are given; let the y-intercept be b and the x-intercept a.

Here the line is determined by two points $(a, 0)$ and $(0, b)$. We have at once,



$$m = -\frac{b}{a}.$$

Then by (7)

$$y - b = -\frac{b}{a}(x - 0),$$

which reduces to

$$\frac{x}{a} + \frac{y}{b} = 1. \tag{9}$$

This is called the *intercept form.* It is used in writing the equation of a straight line when the intercepts are known.

32. Lines Parallel to the Axes. — The equation of a line parallel to the y-axis cannot be written in forms (8) or (9), since there is no y-intercept. The equation of such a line is obviously

$$x = a.$$

Similarly the equation of a line parallel to the x-axis has the form

$$y = b.$$

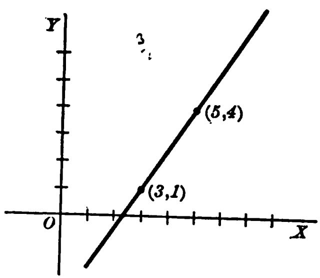
Exercise 1. Derive the intercept form geometrically* when

(*a*) a is positive and b negative;
(*b*) a is negative and b positive;
(*c*) both a and b are negative.

Exercise 2. Derive the slope intercept form geometrically* for the cases given in Exercise 1.

PROBLEMS

1. Derive the equation of the line determined by the points (3, 1) and (5, 4). Also reduce the result to the slope intercept form.



Solution. — By the slope formula $m = \frac{3}{2}$. Using the point slope form,

$$y - 1 = \tfrac{3}{2}(x - 3),$$

which reduces to

$$3x - 2y - 7 = 0.$$

To reduce this to the slope intercept form, solve for y:

$$y = \tfrac{3}{2}x - \tfrac{7}{2}.$$

This form shows that the slope is $\frac{3}{2}$ and the y-intercept is $-\frac{7}{2}$.

Equations can be readily checked by substituting the coördinates of known points or drawing the line from the intercepts and observing whether or not it satisfies the given conditions.

2. Derive the equation of the line determined by the given points in each of the following, and find its slope and intercepts.

(*a*) $(-2, 2), (6, 5)$; (*c*) $(7, 5), (-4, -3)$;
(*b*) $(4, 2), (6, -5)$; (*d*) $(8, -5), (-4, 3)$.

* *I.e.* without using the slope formula.

3. Derive the equation of the line

(*a*) Passing through (3, 5) and of slope $\frac{3}{2}$;
(*b*) Passing through $(-5, 2)$ and of slope $-\frac{1}{2}$;
(*c*) Passing through $(-5, -1)$ and of slope 2;
(*d*) Passing through (0, 0) and of slope $-\frac{3}{2}$;
(*e*) Passing through (5, 1) and parallel to the y-axis;
(*f*) Passing through (1, 5) and parallel to the x-axis;
(*g*) Passing through (8, 2) and of inclination 30°;
(*h*) Passing through $(-2, 5)$ and of inclination 135°;
(*i*) Passing through (1, 5) and of inclination 90°.

Ans. (*a*) $3x - 2y + 1 = 0$; (*g*) $x - \sqrt{3}y = 8 - 2\sqrt{3}$.

4. Derive the equations of the lines from the given data.

(*a*) Intercepts 2 and -6;
(*b*) Intercepts -6 and $-\frac{3}{2}$;
(*c*) Intercepts $\frac{1}{4}$ and $\frac{1}{3}$;
(*d*) Intercepts $-\frac{3}{8}$ and 2;
(*e*) $m = \frac{1}{2}$, $b = -6$;
(*f*) $m = -2$, $b = 3$;
(*g*) $m = 3$, $a = 4$;
(*h*) $m = -2$, $a = -3$;
(*i*) $a = -3$, $\alpha = \frac{\pi}{6}$;
(*j*) $a = -4$, $\alpha = \frac{2\pi}{3}$;
(*k*) $b = -5$, $\alpha = \pi$;
(*l*) $b = 6$, $\alpha = \frac{3\pi}{4}$.

Ans. (*a*) $6x - 2y - 12 = 0$; (*i*) $x - \sqrt{3}y + 3 = 0$.

5. Find the angle between the line* $2x - 3y = 5$ and each of the lines in Problem 4. *Ans.* (*a*) 37° 52′.

6. Reduce each of the following straight line equations to the intercept and slope intercept forms if possible, and draw the lines.

(*a*) $3x - 4y - 8 = 0$;
(*b*) $\frac{x}{3} - \frac{y}{5} = 1$;
(*c*) $3x - 2y = 6$;
(*d*) $3y + 2 = 0$;
(*e*) $4x - 3y = 0$;
(*f*) $5x +$ [illegible] $y = 4$;
(*g*) $2x - 3y + 3 = 0$;
(*h*) $x - 3y = 0$.

7. Find the equation of the line passing through the point (1, 3) and forming with the axes a triangle of area 8.

8. Show that all lines for which $b = a$ have the same slope.

9. The distance between two fixed points is $2c$. Find the equation of the locus of a point moving so that its distances from the fixed points are equal.

* More properly we should say "the line whose equation is $2x - 3y = 5$," but the close relation between a curve and its equation has given rise to the custom of using the abbreviated phrase.

10. The distance between two fixed points is $2c$. Find the equation of the locus of a point moving so that the difference of the squares of its distances from the fixed points is k.

11. The base of a triangle is fixed in length and position. Find the locus of the opposite vertex if the slope of one side is twice the slope of the other and (*a*) the base lies on the x-axis;
(*b*) the base lies on the y-axis.

12. Show analytically that the centers of circles passing through two fixed points lie on a straight line.

13. Derive the straight line equation $x = ny + a$, where $n = \cot \alpha$, and a is the x-intercept.

14. Find the equations of the two lines through the origin which trisect the triangle formed by the line $\frac{x}{4} + \frac{y}{3} = 1$ and the coördinate axes.

15. Two straight lines of slope -1 are tangent to a circle whose center is the origin and whose radius is 4. Find the equations of the lines.

16. Find the equation of a straight line through the point (4, 3) and having equal intercepts.

33. The Linear Equation. — THEOREM I. *Every straight line is defined by an equation of the first degree, in one or two variables.*

PROOF. It has been shown that every straight line cutting the y-axis may be defined by the equation $y = mx + b$. In case the line is parallel to the y-axis, its equation is of the form $x = a$. Both of these are of the first degree.

THEOREM II. Conversely, *the locus of every first degree equation in one or two variables is a straight line.*

PROOF. The general form of the equation of the first degree is

$$Ax + By + C = 0, \tag{10}$$

where A, B, and C may have any values, zero included.

When $B \neq 0$, this may be solved for y, giving

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

This is of the form $y = mx + b$ and therefore defines a straight line with slope $\frac{-A}{B}$ and y-intercept $\frac{-C}{B}$.

When $B = 0$, we have $\quad x = -\frac{C}{A}$.

This is of the form $x = a$ and therefore defines a straight line parallel to the y-axis and with x-intercept $\frac{-C}{A}$. Thus the theorem is proved for both cases.

NOTE. — For this reason an equation of the first degree is called a linear equation, the term being used by analogy for all equations of the first degree in any number of variables.

Exercise 3. Transform the general equation $Ax + By + C = 0$ into the intercept form. When is this impossible?

34. Relations between Linear Equations. — THEOREM I. *If in two linear equations in two variables the coefficients of the variables are proportional, the lines defined by those equations are parallel,* and conversely.

PROOF. Let the equations be

$$Ax + By + C = 0,$$

and

$$A'x + B'y + C' = 0.$$

Solving for y,

$$y = -\frac{A}{B}x - \frac{C}{B} \text{ and } y = -\frac{A'}{B'}x - \frac{C'}{B'}.$$

These are of the form $\quad y = mx + b.$

Hence $\quad m = -\frac{A}{B} \quad$ and $\quad m' = -\frac{A'}{B'}.$

But by hypothesis $\quad A : A' = B : B'$ or $A : B = A' : B'$.

Thus we have $m = m'$, and the lines having equal slopes are parallel.

The converse and the special case where $B = B' = 0$ are left to the student.

Exercise 4. Assume the lines parallel and prove that the coefficients of x and y are proportional.

THEOREM II. *In any two linear equations,*

$$Ax + By + C = 0,$$
$$A'x + B'y + C' = 0,$$

if $AA' = -BB'$, the lines defined by the equations are perpendicular, and conversely.

PROOF. Solving the given linear equations for y, as before, we have

$$m = -\frac{A}{B} \quad \text{and } m' = -\frac{A'}{B'}.$$

Then $$mm' = \left(-\frac{A}{B}\right)\left(-\frac{A'}{B'}\right) = \frac{AA'}{BB'}.$$

But by hypothesis $AA' = -BB'$, hence $mm' = -1$, and the condition of perpendicularity is satisfied.

Exercise 5. Assume the lines perpendicular and prove that

$$AA' = -BB'.$$

THEOREM III. *If in any two linear equations*

$$Ax + By + C = 0,$$
$$A'x + B'y + C' = 0,$$

we have $A : A' = B : B' = C : C'$, the lines defined by the equations are identical, and conversely.

PROOF. From the assumed proportions

$$\frac{C}{B} = \frac{C'}{B'}, \text{ or } b = b'.$$

Also $$\frac{A}{B} = \frac{A'}{B'}, \text{ or } m = m' \text{ as before.}$$

Hence the lines have the point $(0, b)$ in common and also have the same direction. They are therefore identical.

Exercise 6. Assume the lines identical and prove that

$$A : A' = B : B' = C : C'.$$

With the aid of these theorems it is possible to determine by inspection of their equations whether two lines are iden-

tical, parallel, perpendicular, or otherwise. This is of considerable aid in checking the solutions of problems involving straight lines. Suppose, for example, that it is desired to write the equation of a line passing through the point $(4, -2)$ and perpendicular to the line $10x - 4y + 11 = 0$.

From the given equation the slope of the line is $\frac{5}{2}$, and hence the slope of the required line is $-\frac{2}{5}$ (§ 12). Substituting in the point slope equation and reducing, we get $2x + 5y + 2 = 0$. This line is evidently perpendicular to the given line, for $AA' = 20$ and $BB' = -20$. Also it passes through the given point, for $2 \cdot 4 + 5 \cdot -2 + 2 = 0$.

The same problem may also be solved by using Theorem II. Any line having the form $4x + 10y = k$ will be perpendicular to the given line $10x - 4y + 11 = 0$. Since the line passes through $(4, -2)$,

$$k = 4 \cdot 4 + 10(-2) = 16 - 20 = -4.$$

Hence the required equation is

$$4x + 10y = -4, \text{ or } 2x + 5y + 2 = 0.$$

35. Application to the Solution of Linear Equations. — Two equations of the first degree in two variables are classified as

(*a*) *Simultaneous* if they have one solution, which is usually the case;

(*b*) *Incompatible* if they have no common solution, as

$$\begin{cases} 3x + 7y = 1, \\ 6x + 14y = 1; \end{cases}$$

(*c*) *Dependent* if they have innumerable solutions, as

$$\begin{cases} 9x - 6y = 3, \\ 3x - 2y = 1. \end{cases}$$

It is sometimes convenient to determine the number of solutions without solving. The test by which the class of the equations can be determined is easily deduced from the relation between their loci, and from Theorems I and III, § 34.

(*a*) The equations have one and only one solution, if their lines intersect. This is the case when

$$A : A' \neq^{*} B : B'.$$

(*b*) The equations have no common solution, if their lines are parallel. This is the case when

$$A : A' = B : B'.$$

(*c*) The equations have innumerable solutions, if their lines have more than one point in common and are therefore identical. This is the case when

$$A : A' = B : B' = C : C'.$$

PROBLEMS

1. Select two pairs each of parallel and perpendicular lines from the following:

(*a*) $y = 2x + 3$;
(*b*) $2y = -6x + 9$;
(*c*) $y = 2x + 9$;
(*d*) $2y = -\frac{1}{2}x + 7$;
(*e*) $x + 5y = 0$;
(*f*) $6x + 2y + 1 = 0$;
(*g*) $5x - y - 8 = 0$;
(*h*) $x + 2y - 7 = 0$.

2. Identify the following pairs of equations as simultaneous, incompatible, or dependent:

(*a*) $2x + 6y - 7 = 0$, $6x + 18y + 8 = 0$; (*b*) $2x + 5y = 3$, $2x - 5y = 8$; (*c*) $x - 3y = 8$, $12y - 4x = -32$.

3. Show that the equations $2Ax + By + D = 0$ and $Bx + 2Cy + F = 0$ are simultaneous if $B^2 - 4AC \neq 0$.

4. Find the equations defining the lines determined as specified and verify the result. In each case two lines are to be found.

(*a*) Both of the required lines pass through (3, 0) and are respectively parallel and perpendicular to $6x - 5y + 10 = 0$;

(*b*) As before, using $(-2, 6)$ and $8x + 7y + 12 = 0$;
(*c*) As before, using (4, 3) and $3x + 2y = 3$;
(*d*) As before, using (0, 0) and $2x - 5y = 6$;
(*e*) As before, using (2, 0) and $3x + 2y = 15$;
(*f*) As before, using (0, 0) and $5x + 5y = 12$;
(*g*) As before, using (3, 3) and $3x - 2y = 10$;
(*h*) As before, using (0, 0) and $x + 6 = 0$.

Ans. (*a*) $6x - 5y - 18 = 0$; $5x + 6y - 15 = 0$.

* This symbol has the meaning, "is unequal to."

5. Show that the bisectors of the angles of any rectangle form a square.

6. In the triangle whose vertices are (0, 0), (6, 0), and (8, 6) find

(*a*) the equations of its sides;

(*b*) the equations of the perpendicular bisectors of the sides;

(*c*) the equations of the perpendiculars from the vertices on the opposite sides;

(*d*) the equations of the medians.

7. Show that in the above problem

(*a*) the perpendicular bisectors meet in a point;

(*b*) the medians meet in a point;

(*c*) the perpendiculars from the vertices on the opposite sides meet in a point.

(In each case show that the coördinates of the intersection of two lines satisfy the equation of the third.)

8. Show that the three points of intersection found in the above problem lie in a straight line.

9. Show that the medians of any triangle meet in a point. (Take the vertices as (0, 0) $(a, 0)$, and (b, c).)

10. Show that the perpendicular bisectors of the sides of a triangle meet in a point.

11. Show that the perpendiculars from the vertices of a triangle on the opposite sides meet in a point.

12. Show that the three points of intersection found in Problems 9, 10, and 11 lie in a straight line.

13. The vertices of a parallelogram are (0, 0), (6, 0), (4, 8), and (10, 8). Lines are drawn from two opposite vertices to the mid-points of opposite sides. Show that they are parallel and that they trisect one of the diagonals.

14. Find the center and radius of a circle circumscribed about the triangle whose vertices are

(*a*) (0, 0), (− 6, 0), (8, 4);
(*b*) (4, 0), (− 4, 0), (6, 8).

Ans. (*a*) center (− 3, 16), radius $\sqrt{265}$.

15. Two opposite vertices of a square are (− 2, 3) and (4, − 5). Find the other vertices and the equations of its sides.

16. Find the fourth vertex of the parallelogram which has three of its vertices:

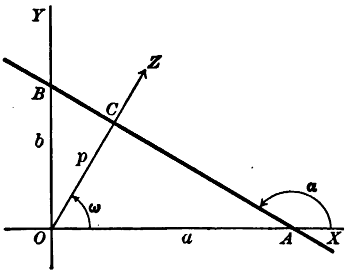
(*a*) (3, 5), (4, 9), (− 2, 6); (*b*) (8, 6), (5, −4), (0, 3).

17. Find the equation of a straight line through the origin making an angle of 45° with each of the lines of Problem 1.

Hint. — To find the slope of the required line, observe that its inclination is 45° greater or less than that of the given line.

Ans. (*a*) $y = -3x$ or $y = \frac{1}{3}x$.

36. Geometric Conditions Determining a Line. — In each of the standard equations $y = mx + b$ and $\frac{x}{a} + \frac{y}{b} = 1$, we note that there are involved two arbitrary constants which represent geometric conditions determining the line. Two of these, the intercepts a and b, represent distances, while the other, the slope m, represents a direction. The direction may also be given by the inclination α.

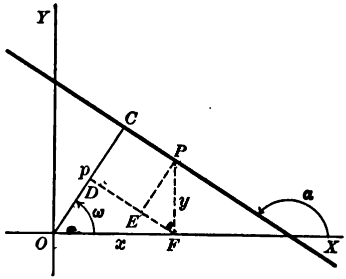


There are also two other such constants obtained as follows. Let AB be the given line and OZ a line from the origin perpendicular to it at C. This is called the *normal axis* of the line. Its inclination is denoted by ω, and the perpendicular distance OC of the line from the origin is denoted by p and called the *normal intercept*.

Thus we associate with any straight line six constants; three, a, b, and p, representing distances, and three, m, α, and ω, representing directions. Any pair of them (one being a distance) will determine the line (and also the other constants), and combined with its current cöordinates they will furnish an equation of the line. Of the many possible equations involving these constants only three are commonly used, two of which have been discussed in §§ 30, 31.

37. The Normal Form. — In this form of the equation the line is determined by p and ω.

To derive it, take any point P on the line and draw FD from the foot of the ordinate of P perpendicular to the normal axis at D; also draw PE perpendicular to FD. It follows at once that $\angle EFP = \omega$,



$$\cos\omega = \frac{OD}{x},$$

and $$\sin\omega = \frac{EP}{y}.$$

But

$$OD + EP = p.$$

Substituting, $$x\cos\omega + y\sin\omega - p = 0. \qquad (11)$$

This is the normal equation. Here ω, the inclination of the normal axis, like α, the inclination of the line, is reckoned from 0° to 180°. The upward direction of p, reckoned from the origin, is taken as positive, and the downward direction, reckoned from the origin, is taken as negative.

38. Reduction of a Linear Equation to the Normal Form. — Let the equation of the given line l be

$$Ax + By + C = 0.$$

It is required to find the equation of l in the normal form.

Since the slope of the given line is $-\frac{A}{B}$, the slope of the normal axis is the negative reciprocal of this, $\frac{B}{A}$. Hence $\tan\omega = \frac{B}{A}$. Elementary trigonometry gives us at once

$$\sin\omega = \frac{B}{\pm\sqrt{A^2+B^2}}, \quad \cos\omega = \frac{A}{\pm\sqrt{A^2+B^2}}.$$

Since $\omega < 180°$, $\sin\omega$ is positive. Therefore the sign of the radical must be taken to agree with that of B. Thus we have determined $\sin\omega$ and $\cos\omega$ in terms of the coefficients of the given equations.

Dividing the given equation by $\sqrt{A^2+B^2}$, we have

$$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y+\frac{C}{\sqrt{A^2+B^2}}=0,$$

which is of the form

$$x\cos\omega+y\sin\omega-p=0.$$

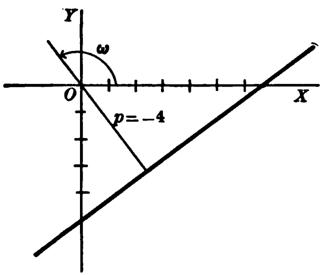
Comparing these equations, we see that p must be

$$\frac{-C}{\sqrt{A^2+B^2}}.$$

The special cases where $A=0$ and $B=0$ are left to the student.

Summing up, we have the following working rule:

To reduce the linear equation $Ax+By+C=0$ to the normal form, divide it by $\pm\sqrt{A^2+B^2}$, giving the radical the sign of B.



Example. — Reduce $3x-4y=20$ to the normal form.

Here $\sqrt{A^2+B^2}=5$, and B is negative. Therefore the required equation is

$$-\frac{3}{5}x+\frac{4}{5}y+4=0.$$

This gives $\cos\omega=-\frac{3}{5}$, $\sin\omega=\frac{4}{5}$, and $p=-4$. Plotting the equation affords an excellent check on the work.

The equation $Ax+By+C=0$ can always be reduced to the normal form. For the division by the radical is always possible, since A and B cannot both be zero. This constitutes one of the two most important advantages of the normal form. The other consists in its convenience for use in finding the distance from a line to a point.

Exercise 7. Derive the normal equation for a line crossing the axes

(*a*) in the second quadrant;
(*b*) in the third quadrant;
(*c*) in the fourth quadrant.

PROBLEMS

1. Draw the lines satisfying the following conditions:

(*a*) $a = -5,\ \alpha = 30°$;
(*b*) $b = 4,\ \alpha = 60°$;
(*c*) $p = -3,\ \alpha = 120°$;
(*d*) $a = 5,\ \omega = 150°$;
(*e*) $b = -4,\ \omega = 120°$;
(*f*) $p = 3,\ \omega = 45°$;
(*g*) $a = -5,\ p = 3$;
(*h*) $b = -5,\ p = -4$;
(*i*) $a = -3,\ p = -3$.

2. With *ruler* and *compasses* construct the required line, given

(*a*) a and α;
(*b*) b and α;
(*c*) p and α;
(*d*) a and ω;
(*e*) b and ω;
(*f*) p and ω;
(*g*) a and p;
(*h*) b and p.

3. Write the equations of the lines of Problem 1.

Ans. (*a*) $x - \sqrt{3}\,y + 5 = 0$; (*g*) $3x - 4y + 15 = 0$.

4. Find the distance of a line from the origin when the intercepts are:

(*a*) 3 and 4;
(*b*) 3 and -4;
(*c*) -6 and 8;
(*d*) 6 and 9;
(*e*) 5 and -12;
(*f*) -5 and 12;
(*g*) 7 and 2;
(*h*) 7 and $\sqrt{15}$.

Ans. (*a*) $\frac{12}{5}$.

5. Find the equations of the lines from the following data:

(*a*) $p = -6,\ \omega = 45°$ or $135°$;
(*b*) $p = 5,\ \omega = 150°$;
(*c*) $p = 4,\ \omega = 60°$ or $120°$;
(*d*) $p = 3,\ \omega = 0°,\ 90°$, or $180°$;
(*e*) $p = 6,\ \omega = 75°$. $\left(\sin 75° = \cos 15° = \sqrt{\dfrac{1 + \cos 30°}{2}}.\right)$

6. Write the equation of the line passing through the point $(5, -10)$ and having

(*a*) $\omega = 135°$;
(*b*) $\omega = 120°$;
(*c*) $p = -2$;
(*d*) $p = -6$.

Ans. (*a*) $x - y - 15 = 0$; (*c*) $4x + 3y + 10 = 0$.

7. Reduce to the normal form and verify the work by constructing the line from its intercepts in each of the following cases:

(*a*) $4x - 3y = 12$;
(*b*) $8x + 15y = 12$;
(*c*) $3x + 4y = 12$;
(*d*) $12x + 5y = 0$;
(*e*) $x - y = 0$;
(*f*) $x = \sqrt{2} + y$;
(*g*) $x + \sqrt{3}\,y = 10$;
(*h*) $6x - 8y = 25$;
(*i*) $x + 2y + 5 = 0$.

8. Find p and $\tan \omega$ in terms of a and b.

Ans. $p = \dfrac{ab}{\sqrt{a^2 + b^2}}$; $\tan \omega = \dfrac{a}{b}$.

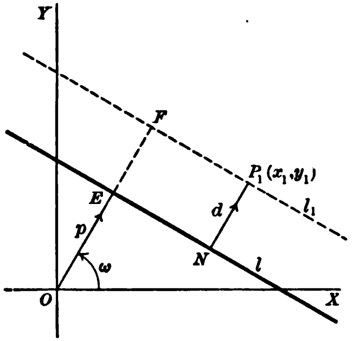
9. Prove $\tan \omega = -\cot \alpha$.

10. Find $\tan \omega$ in terms of A and B.

11. Derive the intercept form of the straight line equation from the normal form.

12. Find the coördinates of the foot of the perpendicular from the origin to a line in terms of its intercepts and inclination.

39. Distance from a Line to a Point. — Let l be a line of which the equation is given and P_1 a point whose coördinates (x_1, y_1) are given, and let d be the distance NP_1 from the line to the point, *reckoned from the line.*



Through P_1 draw the line l_1 parallel to l and, if necessary, produce the normal p to meet it. The normal inclination ω is the same for l and l_1, and is known from the equation of l when it is reduced to the normal form. The normal intercept of l_1 is OF and

$$OF = OE + EF = p + d.$$

Hence the normal equation of l_1 is

$$p + d = x \cos \omega + y \sin \omega.$$

This is satisfied by the coördinates of P_1; hence

$$p + d = x_1 \cos \omega + y_1 \sin \omega, \text{ or}$$

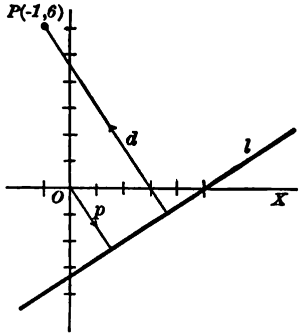
$$\mathbf{d = x_1 \cos \omega + y_1 \sin \omega - p.} \qquad (12)$$

But the equation of l is

$$x \cos \omega + y \sin \omega - p = 0. \qquad (11)$$

Thus the first member of (11) becomes the value of d when the coördinates of P_1 are substituted for x and y. Hence follows the rule:

To find the distance from a line to a point, reduce the equation of the line to the normal form and substitute the coördinates of the given point for the variables. The value of the first member is the required distance.



EXAMPLE. — Find the distance of the point $P(-1, 6)$ from the line $2x - 3y = 10$.

The normal equation of l is

$$-\frac{2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}} + \frac{10}{\sqrt{13}} = 0.$$

$$\therefore d = \frac{(-2)(-1)}{\sqrt{13}} + \frac{3 \cdot 6}{\sqrt{13}} + \frac{10}{\sqrt{13}}$$
$$= \frac{30}{\sqrt{13}}.$$

Here p, being negative, is reckoned downward from the origin and d, being positive, is reckoned upward from the line l.

Exercise 8. Show that the distance formula can be written

$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$$

where the radical should have the sign of B.

PROBLEMS

1. In each case find the distance from the line to the point, and verify results by constructing a figure.

(a) $3x - 4y = 10$, $(2, 1)$; (d) $3x + 4y = 12$, $(4, 0)$;
(b) $x = y$, $(0, 4)$; (e) $5x + 12y = 5$, $(0, 0)$;
(c) $4x - 2y + 5 = 0$, $(-3, 6)$; (f) $x + y + 1 = 0$, $(4, 4)$.

2. Find the distance between the parallel lines

(a) $y = 3x + 5$, $y = 3x - 5$; (b) $3x + 2y = 2$, $3x + 2y = 6$; (c) $6x + 9y = 0$, $4x + 6y = -8$.

Ans. (a) $\sqrt{10}$.

3. Find the distance from the intersection of the lines

$$(a)\quad \begin{aligned} x - 4y - 12 &= 0 \\ 3x - 4y - 2 &= 0 \end{aligned} \quad \text{to the line} \quad x + y + 10 = 0;$$

$$(b)\quad \begin{aligned} 3x - y - 2 &= 0 \\ 5x + y - 12 &= 0 \end{aligned} \quad \text{to the line} \quad 12x - 5y + 23 = 0.$$

4. Find the areas of the triangles formed by the three lines in (*a*) and (*b*) in Problem 3. *Ans.* (*a*) $\frac{9}{140}$.

5. Find the areas of the triangles whose vertices are

(*a*) $(0, 0)$, $(1, 2)$, $(5, 7)$; (*c*) $(-7, 8)$, $(-7, -6)$, $(1, -4)$;
(*b*) $(-1, 3)$, $(2, -4)$, $(0, 6)$; (*d*) $(5, 4)$, $(8, 3)$, $(6, 5)$.

Ans. (*a*) $\frac{3}{2}$; (*c*) 56.

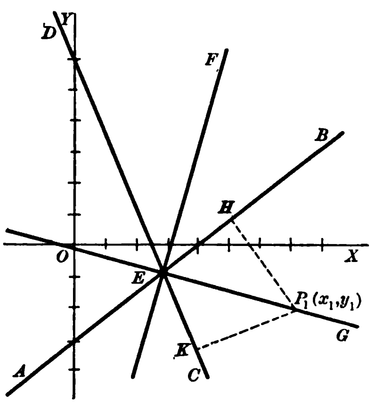
6. Find correct to one decimal place the areas of the polygons having the following vertices:

(*a*) $(6, 0)$, $(7, 10)$, $(-3, 5)$, $(-2, -6)$;
(*b*) $(2, 0)$, $(1, 4)$, $(-2, 2)$, $(-1, -3)$, $(1, -2)$.

Ans. (*a*) 94.5; (*b*) 17.5.

7. Find the equations of the bisectors of the angles between the lines $3x - 4y = 12$ and $12x + 5y = 30$.

Solution. — Let the lines meet at E; there are two bisectors EF and EG. Let $P_1(x_1, y_1)$ lie upon EG, the bisector of the angle BEC. Then by plane geometry the perpendiculars HP_1 and KP_1 are of equal length.



To find the length of HP_1 put $3x - 4y = 12$ into the normal form. Then

$$HP_1 = -\frac{3x_1}{5} + \frac{4y_1}{5} + \frac{12}{5}.$$

Similarly

$$KP_1 = \frac{12x_1}{13} + \frac{5y_1}{13} - \frac{30}{13}.$$

But HP_1 is negative and KP_1 positive. Therefore

$$HP_1 = -KP_1, \text{ or}$$

$$-\frac{3x_1}{5} + \frac{4y_1}{5} + \frac{12}{5} = -\frac{12x_1}{13} - \frac{5y_1}{13} + \frac{30}{13}.$$

Simplifying and dropping primes, $21x + 77y + 6 = 0$ is the equation of EG.

In like manner the equation of EF is

$$99x - 27y - 306 = 0.$$

A convenient check on the accuracy of the work is the application of the test for perpendicularity to the two solutions.

8. Find the equations of the bisectors of the angles between

$$(a)\ 3x + 4y = 5,\quad 4x + 3y = 6; \qquad (b)\ 3x - 4y = 7,\quad 4x - 3y = 8.$$

9. Find the locus of points twice as far numerically from the first line as from the second in each of the groups in Problem 8.

Ans. (a) $5x + 2y - 7 = 0$, $11x + 10y - 17 = 0$.

10. Find the locus of points the sum of whose distances from each of the pairs of lines in Problem 8 is 12.

Ans. (a) $7x + 7y - 71 = 0$.

11. Find the locus of points 6 units distant from the line

(a) $3x + 4y = 5$; (c) $15x - 8y = 12$.
(b) $12x - 5y = 17$;

Ans. (a) $3x + 4y = 35$, $3x + 4y = -25$.

12. A triangle has vertices (3, 0) and (– 1, 3). Find the locus of the third vertex if the area is to be 16.

13. Find the equation of the line halfway between the parallels

$$(a)\ 2x + y = 7,\quad 2x + y = 10; \qquad (b)\ 3x - 2y = 9,\quad 3x - 2y = 16.$$

14. A triangle has the vertices (0, 0), ($12\frac{1}{2}$, 0), and (8, 6).

(a) Find the equations of the bisectors of the angles.
(b) Show that these bisectors meet in a point.
(c) Find the coördinates of the center and the radius of the inscribed circle.

40. Systems of Straight Lines. — When one of the constants of a straight line equation is arbitrary and the other absolute, as in $y = 3x + k$, the equation defines a line for each value assigned to the arbitrary constant, but each line so defined has the property given by the numerical constant.

Thus in the above equation k is the y-intercept, and it may have any value; but the slope of each line is 3, and by assigning all possible values to k we have all possible lines of slope 3. Such a group of lines is called a *system*, and the arbitrary constant is called the *parameter* of the system.

We have already seen that any of the standard forms of the straight line equation involves two arbitrary constants

which correspond to the geometric conditions defining the line. Hence if one of these constants is given a numerical value and the other left arbitrary, we get a *system of straight lines* characterized by the geometrical property defined by the numerical constant, and with one geometrical property left arbitrary or variable. This correspondence is general, as will be seen later in the case of the circle and the conics.

PROBLEMS

In each of the following problems draw several lines of the system.

1. Write the equations of the systems of lines defined as follows:

(*a*) passing through $(5, -1)$; *Ans.* $y + 1 = m(x - 5)$.
(*b*) passing through $(-2, 2)$;
(*c*) passing through $(6, -3)$;
(*d*) with the y-intercept -2;
(*e*) with the x-intercept -3;
(*f*) having the slope $-\frac{2}{3}$;
(*g*) having the slope $\frac{1}{2}$;
(*h*) distance from origin 13;
(*i*) distance from origin 5.

2. What geometric conditions define the systems of lines represented by the following equations:

Hint. — Put each equation in a standard form.

(*a*) $x + k = 0$;
(*b*) $3x + 4y + k = 0$;
(*c*) $x + 3ky - 5 = 0$;
(*d*) $kx + 2y - 6 = 0$;
(*e*) $2x - ky + 2 = 0$;
(*f*) $kx + ky - 6 = 0$;
(*g*) $kx + y\sqrt{1 - k^2} = 6$;
(*h*) $y - 1 = mx + 2m$.

Ans. (*c*) x-intercept 5; (*g*) normal 6.

3. Determine k so that

(*a*) the line $3x + 4y = k$ passes through $(-2, 1)$;
(*b*) the line $3x - 4y = k$ passes through $(-3, 4)$;
(*c*) the line $kx - 4y = -3$ has the slope 2;
(*d*) the line $3x - 4y - k = 0$ has its x-intercept $= 3$;
(*e*) the line $3x - 4ky + 5 = 0$ has its y-intercept $= -3$;
(*f*) the line $4x - 3y + k = 0$ is distant 5 units from the origin.

Ans. (*a*) $k = -2$; (*c*) $k = 8$; (*e*) $k = -\frac{5}{12}$.

4. Write the equation of the system of lines perpendicular to

$$y = mx + b.$$

5. Find the equation of the system of lines whose normal axis has the slope $-\frac{5}{12}$.

6. Find the equation of the system of lines whose normal axis has the inclination $\frac{5\pi}{6}$.

7. What is the equation of the system of lines tangent to a circle having the center at the origin and the radius 10?

41. Systems of Lines through the Intersection of Two Given Lines. — Suppose it is desired to find the equation of the system of lines through the intersection of two given lines. One way of doing this would be to solve the two given equations for the point of intersection and to use the point slope equation. A method more expeditious and of great theoretical importance follows.

THEOREM. *Let $l_1 = 0$ and $l_2 = 0$ be the equations* of two intersecting lines l_1 and l_2, and let k be an arbitrary constant. Then $l_1 + k \cdot l_2 = 0$ defines a system of lines passing through the intersection of the given lines.*

PROOF. We first observe that $l_1 + k \cdot l_2 = 0$ defines a system of straight lines with k as a parameter, since it is obtained by adding two first degree equations.

Let the point of intersection P have coördinates (a, b). By the locus definition the equations $l_1 = 0$ and $l_2 = 0$ are both satisfied by $x = a$ and $y = b$, since $P(a, b)$ lies on both curves. That is, the quantities $l_1(a, b)$ and $l_2(a, b)$ are both zero. Therefore $l_1(a, b) + k \cdot l_2(a, b) = 0$ for all values of k.

Hence every line of the system $l_1 + k \cdot l_2 = 0$ passes through P, since the coördinates of P satisfy the equation.†

* The equation $l_1 = 0$ stands for an expression of the form

$$Ax + By + C = 0,$$

and would more properly be written $l_1(x, y) = 0$, but the variables are omitted for convenience.

† We can also prove that *every* line passing through P belongs to the system. Let $l_3 = 0$ be another line through P, and let $P_1(x_1, y_1)$ be some other point on l_3. If the coördinates of P_1 be substituted in $l_1 + k \cdot l_2 = 0$, we obtain an equation

$$l_1(x_1, y_1) + k \cdot l_2(x_1, y_1) = 0$$

in which k, being the only unknown, can be found. For this value of k, the locus of the equation $l_1 + k \cdot l_2 = 0$ passes through P_1, since its coördinates make the left hand member equal to zero. As it already passes through P, it must be identical with l_3, having two points in common with it.

Example. — Find the line containing the point (2, 3) and the intersection of $l_1 : 3x + y - 5 = 0$ and $l_2 : x + 2y - 3 = 0$.

The system $l_1 + k \cdot l_2 = 0$ contains the intersection, and k may be found for the assumed point by substituting the coördinates of that point in the equation of the system, thus:

$$3x + y - 5 + k(x + 2y - 3) = 0$$

becomes $$6 + 3 - 5 + k(2 + 6 - 3) = 0,$$

and $k = -\frac{4}{5}$; from which we have the result $11x - 3y - 13 = 0$.

Verification. Solving the given equations simultaneously, the intersection is $(\frac{7}{5}, \frac{4}{5})$. These coördinates and the coördinates (2, 3) both satisfy the new equation.

The above proof will apply to any two equations of the form $f(x, y) = 0$, as well as to two linear equations. The general principle may be stated as follows:

The equation $f(x, y) + k \cdot g(x, y) = 0$ defines a system of curves passing through the intersections of the curves defined by the two equations $f(x, y) = 0$ and $g(x, y) = 0$.

PROBLEMS

1. Find by the method of the example given above the equation of the system of lines passing through the intersection of $3x - 2y = 16$ and $3x + 6y = 1$. Find the particular line of the system passing through $(-5, 4)$.

2. Find the line of the above system which has the slope $\frac{1}{2}$.

Hint. — The equation in k is $3x - 2y - 16 + k(3x + 6y - 1) = 0$,

or $$(3k + 3)x + (6k - 2)y - k - 16 = 0.$$

Solve this equation for y and the coefficient of x is the slope.

3. Find the line of the above system which is parallel to the x-axis; also the line which is parallel to the y-axis.

4. Find by the method of the illustrative example the equations of the lines determined by

(a) $P(4, 3)$ and the intersection of $4x - 2y = 7$ and $x - 2y = 3$;
(b) $P(3, 0)$ and the intersection of $3x - 2y = -12$ and $3x + 2y = 0$;
(c) $P(-5, 0)$ and the intersection of $3x + 2y = 5$ and $x = 6$;
(d) $P(-4, 1)$ and the intersection of $x - y + 5 = 0$ and $x + y - 5 = 0$.

5. Find by the method of Ex. 2 two lines of the system containing the intersection of the first pair of lines, one parallel, the other perpendicular to the third line:

(a) $3x - 5y + 5 = 0$, $5x + 4y = 10$, and $x + y = 6$;

(*b*) $x + y = 0$, $x - y = 0$, and $2x + 3y + 5 = 0$;
(*c*) $6x - 5y + 4 = 0$, $x + y = 1$, and $x = 3$;
(*d*) $6x - 3y + 9 = 0$, $4x + 3y = 10$ and $5x - 9y + 10 = 0$.

6. Solve Problem 4 (*a*) by finding the point of intersection and using (7).

7. Find and plot the equations of two curves which pass through the intersections of

(*a*) $y^2 = 8x + 16$, $y^2 = 4x + 16$; (*b*) $x^2 + y^2 = 13$, $x^2 - y^2 = 5$.

42. Plotting by Factoring. — THEOREM. *If $l_1 = 0$ and $l_2 = 0$ define the lines l_1 and l_2, the locus of $l_1 \cdot l_2 = 0$ consists of these lines.*

PROOF. This follows directly from the locus definition. For the coördinates of a point (a, b) on either line, say l_1, satisfy its equation $l_1 = 0$. But if $l_1(x, y)$ vanishes for $x = a$ and $y = b$, the product of $l_1(x, y)$ and $l_2(x, y)$ must vanish for the same values, *i.e.* (a, b) lies on the locus of $l_1 \cdot l_2 = 0$.

Conversely, if (a, b) is on this locus, one of the factors $l_1(x, y)$, $l_2(x, y)$ must vanish for $x = a$, $y = b$. Hence (a, b) lies on the line defined by the equation formed by setting this factor equal to zero.

This proof is made clearer by application to a particular example.

Consider the equation

$$x^2 - 4y^2 - x - 2y = 0.$$

Factoring, we have

$$(x + 2y)(x - 2y - 1) = 0.$$

Now the coördinates of any point on the line $x + 2y = 0$, as $(2, -1)$ make the first factor zero and hence the product is zero. Therefore all points on $x + 2y = 0$ lie on the locus of the given equation. The same is true of $x - 2y - 1 = 0$. Conversely, if a point lies on the locus of $x^2 - 4y^2 - x - 2y = 0$, its coördinates satisfy the equation and must make at least one of the factors of the left-hand member zero. Therefore

all points of the given locus belong to one of the lines $x+2y=0$, $x-2y-1=0$.

This reasoning applies to all equations of the form $f(x,y)=0$, whether they define straight lines or not, although it is useful chiefly for linear factors. It leads to the following rule.

Rule for Plotting by Factoring. — *Transpose all terms to the first member. Factor as far as possible; set each factor equal to zero, and plot the resulting equations on the same axes.*

PROBLEMS

1. Plot by factoring the pairs of lines defined by the following equations:

(*a*) $x^2-9y^2=0$;
(*b*) $x^2+xy-2y^2+3x-3y=0$;
(*c*) $x^3-3x^2y-4xy^2=0$;
(*d*) $x^2-x-y^2+y=0$;
(*e*) $4x^2+4xy+y^2+2x+y-2=0$;
(*f*) $x^2+2x+1-y^2+6y-9=0$;
(*g*) $y^2+3y+2=0$;
(*h*) $x^2-4y^2-x-2y=0$.

2. Draw the locus defined by:

(*a*) $5xy^2=7x^2y$; (*b*) $8x^3y-5xy^3=0$.

3. Find the area of the triangle defined by:

(*a*) $x^2y-2xy^2+3xy=0$;
(*b*) $(x+3)(xy-y^2)=0$;
(*c*) $x^3-6x^2-xy^2+6y^2=0$.

4. The sides of a quadrilateral are defined by $y^2-4xy+4x^2-3y+6x=0$ and $x^2+4x+3=0$. Prove that it is a parallelogram.

5. Plot the locus of the equation:

(*a*) $(x^2+y^2-25)(x^2-y^2-25)=0$; (*b*) $x^3y+xy^3=16xy$.

6. Show that $Ax^2+Bx+C=0$ defines a pair of lines parallel to the y-axis if $B^2-4AC>0$, a single line if $B^2-4AC=0$, and an imaginary locus if $B^2-4AC<0$.

7. Show that the locus of $Ax^2+Bxy+Cy^2=0$ is

(*a*) a pair of intersecting lines if $B^2-4AC>0$;
(*b*) a single line if $B^2-4AC=0$;
(*c*) imaginary if $B^2-4AC<0$.

8. Find the equations of the bisectors of the angles between the lines defined by $x^2 - 4y^2 + 3x + 6y = 0$.

9. The sides of a parallelogram are $y^2 - 6y = 0$ and $(y - 2x)^2 + 6(y - 2x) = 0$. Show that the perpendiculars upon a diagonal from opposite vertices are of equal length.

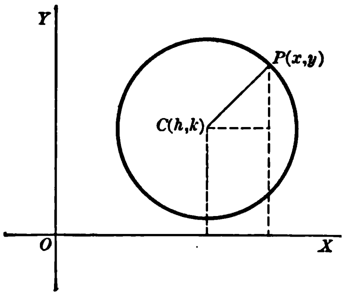
10. Generalize and solve Problem 9.

11. The vertices of a triangle are (0, 0), (5, 0), (0, − 3). Write an equation defining its three sides. *Ans.* $3x^2y - 5xy^2 - 15xy = 0$.

CHAPTER III

THE CIRCLE

43. The Circle Equation. — While the straight line equation is expressed in several standard forms, depending on the choice of the geometric constants determining it, we use only one such form of the circle equation in rectangular coördinates. The constants used in determining the circle are the coördinates of its center and the length of its radius.



Let the coördinates of the center of the circle be (h, k), the length of the radius r, and the coördinates of the point tracing the circle (x, y). By the distance formula,

$$r = \sqrt{(x-h)^2 + (y-k)^2}.$$

This gives

$$(x-h)^2 + (y-k)^2 = r^2 \tag{13}$$

as the standard form of the equation of the circle.

When the center is at the origin, this reduces to the form

$$x^2 + y^2 = r^2. \tag{13 a}$$

PROBLEMS

1. Write the forms which the equation of the circle takes in the following cases:

(*a*) center on the x-axis; (*c*) circle touching the x-axis;
(*b*) center on the y-axis; (*d*) circle touching the y-axis;
(*e*) circle touching both axes.

2. In each of the following cases write the equation of the circle in the standard form and simplify the result; also check the work by finding the intercepts and drawing the circle:

(*a*) center at (3, 5) and radius 5;
(*b*) center at $(-8, -6)$ and radius 10;
(*c*) center at $(4, -3)$ and radius 5;
(*d*) center at $(-8, 6)$ and radius 8;
(*e*) center at (6, 0) and radius 6.

Ans. (*a*) $x^2 + y^2 - 6x - 10y + 9 = 0$.

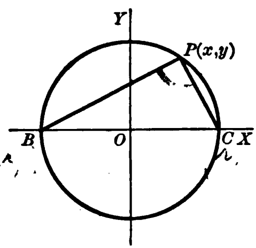
3. Write the equations of the circles satisfying the following conditions and draw the figure in each case:

(*a*) tangent to x axis, radius 3, abscissa of center 6;
(*b*) tangent to y axis, radius 4, ordinate of center -5;
(*c*) tangent to both axes, radius 16;
(*d*) passing through origin, radius 5, abscissa of center -4;
(*e*) passing through origin, radius 13, abscissa of center -5;
(*f*) passing through (2, 2), radius 5, ordinate of center -1;

Ans. (*a*) $x^2 + y^2 - 12x \pm 6y + 36 = 0$;
(*d*) $x^2 + y^2 + 8x \pm 6y = 0$.

4. Prove that an angle inscribed in a semicircle is a right angle.

Solution. — Let the radius of the circle be r; take the diameter as the x-axis and the center as the origin. Let $P\ (x, y)$ be any point on the circumference. We have to prove that BPC is a right angle.



Now the coördinates of B are $(-r, 0)$ and of C, $(r, 0)$. Hence by the slope formula, the slope of BP is $\frac{y}{x+r}$; that of CP is $\frac{y}{x-r}$. Calling these slopes m and m', we have

$$mm' = \frac{y^2}{x^2 - r^2}.$$

But $P(x, y)$ is on the circle. Therefore, $x^2 + y^2 = r^2$, or $y^2 = r^2 - x^2$. This gives

$$mm' = \frac{r^2 - x^2}{x^2 - r^2} = -1,$$

which proves that BP and CP are perpendicular, or that BPC is a right angle.

5. Prove without the use of slopes that an angle inscribed in a semicircle is a right angle.

6. Prove that a perpendicular from any point on the circumference of a circle to a fixed diameter is a mean proportional between the segments into which it divides the diameter.

7. Prove that angles inscribed in the same segment of a circle are equal.

Hint. — Take the chord whose ends are (c, d) and $(-c, d)$ and use the angle formula.

8. Prove that a line from the center of a circle bisecting a chord is perpendicular to it.

Hint. — Let the ends of the chord be $(-r, 0)$ and (b, c).

44. The General Equation of the Circle. — If we expand the standard form (13), we get

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0.$$

This is of the form

$$x^2 + y^2 + Dx + Ey + F = 0, \tag{14}$$

which may be called the *general form* of the circle equation. Conversely, this form represents a circle, since by completing squares it is possible to reduce it back to form (13), which we know is the equation of a circle of center (h, k) and radius r. (Note exception, § 45.)

The general equation of the second degree has the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Comparison with (14) shows that this can be reduced to form (14) if $B = 0$ and $A = C$, by dividing by A. Hence we have the theorem:

An equation of the second degree in two variables defines a circle when, and only when, the coefficients of x^2 and y^2 are equal and the term in xy is missing.

45. Identification of Center and Radius. — To find the center and radius of a circle when the equation is given, reduce the equation to form (14).

Example. — Find the center and radius of the circle

$$2x^2 - 8x + 2y^2 + 6y - 21 = 0.$$

Dividing by 2 and completing the squares, we have

$$x^2 - 4x + 4 + y^2 + 3y + \tfrac{9}{4} = \tfrac{21}{2} + 4 + \tfrac{9}{4},$$

or

$$(x - 2)^2 + (y + \tfrac{3}{2})^2 = \tfrac{67}{4}.$$

This is of form (13) and therefore represents a circle whose center is $(2, -\frac{3}{4})$ and whose radius is $\frac{1}{4}\sqrt{67} = 4.1$ approximately.

Degenerate Forms. — Sometimes on completing the squares the term corresponding to r^2 is negative or zero. In the former case, it may be shown by the methods of § 26 that the locus is imaginary and we say that the graph is an *imaginary circle.* In the latter case, the radius of the circle is zero, and the graph is called a *point circle.* Such forms are sometimes known as degenerate forms.

46. Special Forms. — Comparing the general form (14) and the expansion of the standard form (13) in § 44, we see that

$$D = -2h,\ E = -2k, \text{ and } F = h^2 + k^2 - r^2.$$

The first two of these relations enable us to determine the center by inspection. They also enable us to determine the following special forms:

if $D = 0$, $h = 0$ and the center is on the y-axis;
if $E = 0$, $k = 0$ and the center is on the x-axis;
if $D = 0$ and $E = 0$, the center is at the origin;
if $F = 0$, $h^2 + k^2 = r^2$, and the origin is on the circumference.

Exercise 1. Prove by considerations of symmetry that if $D = 0$, the center is on the y-axis.

PROBLEMS

1. Prove that each equation defines a circle. Also determine the center and radius and draw each circle.

(*a*) $(x+3)^2 + (y-4)^2 = 25$;
(*b*) $(x+4)^2 + (y-5)^2 = 0$;
(*c*) $x^2 + y^2 - 4x + 6y - 3 = 0$;
(*d*) $x^2 + y^2 + 6y = 0$;
(*e*) $15x^2 + 15y^2 = 5x$;
(*f*) $x^2 + y^2 + 2x + 8y + 1 = 0$;
(*g*) $2x^2 + 2y^2 - 16x - 20y = 0$;
(*h*) $x^2 + y^2 + 5x - 3y - \frac{1}{2} = 0$;
(*i*) $x^2 + y^2 - 24x + 10y + 169 = 0$;
(*j*) $x^2 + y^2 + 2x + 2y + 2 = 0$.

2. Determine by inspection which of the circles in Problem 1

(*a*) have their centers on the x-axis;
(*b*) have their centers on the y-axis;
(*c*) pass through the origin.

3. Determine by inspection the centers of the circles in Problem 1.

4. Write the equations of the circles symmetrical to those of Problem 1 with respect to the x-axis and in each case draw both circles.

5. In the system of circles defined by $(x-4)^2+(y-3)^2=k$, discuss the center, radius, and intercepts for

$$k = 25,\ 16,\ 9,\ 4,\ 1,\ 0,\ -1.$$

6. For what values of k do the following equations have a locus?

(*a*) $x^2+y^2+ky=-8$; (*b*) $x^2+y^2+kx=-6$.

Ans. (*a*) $k \geq 4\sqrt{2}$.

7. Find the values of k for which each of the equations of Problem 6 defines a point.

8. Find the points of intersection and the equation of the common chord of the circles:

(*a*) $x^2+y^2+8y=20$, $x^2+y^2-8x=4$; (*b*) $x^2+y^2+4x-60=0$, $x^2+y^2-4y-36=0$.

9. (*a*) The point $(\frac{7}{2}, \frac{1}{2})$ bisects a chord of the circle $x^2+y^2=25$. Find the equation of the chord and its length.

(*b*) The same for $(4, 7)$ and $x^2+y^2-4x-6y-12=0$.

Ans. (*a*) $7x+y-25=0$, $5\sqrt{2}$;
(*b*) $x+2y-18=0$, $2\sqrt{5}$.

10. Find the equation of a circle tangent to the line $4x+3y-7=0$ and having its center at the origin.

11. Find the locus of the vertex of a right triangle which has the ends of its hypothenuse at

(*a*) $(2, 3)$ and $(-2, 6)$; (*b*) $(-4, 3)$ and $(0, 5)$.

General Hint. — In each of the following locus problems, the directions of § 16 as to choice of axes should be followed. After deriving the equation, the locus should be identified and drawn, and its relation to the given data stated.

12. (*a*) The distance between two fixed points is $2c$. Find the locus of a point moving so that the sum of the squares of its distances from the points is $4c^2$.

(*b*) Solve the same problem when the sum of the squares is any constant, as k. *Ans.* The locus is a circle, whose center is midway

between the given points and whose radius is $\sqrt{\frac{k-2c^2}{2}}$. If $k < 2c^2$, there is no locus.

13. The base of a triangle is fixed in length and position. Find the locus of the opposite vertex if

(*a*) the vertical angle is 90°;
(*b*) the vertical angle is 45°;
(*c*) the vertical angle is any constant;
(*d*) the median to one of the variable sides is constant.

14. The ends of a straight line of constant length rest on two perpendicular lines. Find the locus of the middle point.

15. Find the locus of a point whose distance from one fixed point is k times its distance from another.

16. Find the locus of a point if the sum of the squares of its distances from (*a*) the sides, (*b*) the vertices, of a given square is constant.

17. From one end of a diameter of a circle whose radius is r chords are drawn and produced their own length. Find the locus of the ends of these lines.

18. From the point $(-r, 0)$ chords are drawn in the circle $x^2 + y^2 = r^2$. Find the locus of their mid-points.

47. The Equation of the Circle Derived from Three Conditions. — The correspondence between the geometrical conditions necessary to determine a circle and the algebraic conditions necessary to determine its equation are analogous to those discussed for the straight line. The number of independent constants in the general linear equation

$$Ax + By + C = 0$$

is two, for if we divide each term by A, we have the form

$$x + B'y + C' = 0.$$

Geometrically, a straight line is determined by two points, or in general by two geometric conditions. Both forms (13) and (14) of the circle equation involve three arbitrary constants. Geometrically a circle is determined by three points not on a straight line. A circle may be determined in other

ways than by the condition of passing through three points, but the determining condition is in any case threefold.

To derive the equation when the geometric conditions are given, it is necessary to express these conditions in three equations involving h, k, and r (or D, E, and F) and solve them simultaneously. The method is illustrated by the following:

EXAMPLE. — Find the equation of the circle determined by (1, 7), (8, 6), and (7, −1).

Each pair of these coördinates must satisfy the circle equation (13).

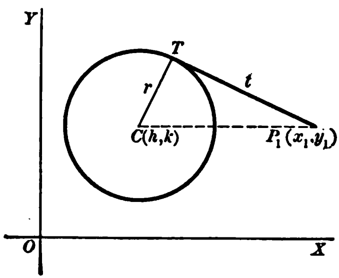
Hence
$$(1-h)^2+(7-k)^2=r^2,$$
$$(8-h)^2+(6-k)^2=r^2,$$
$$(7-h)^2+(-1-k)^2=r^2.$$

Solving these equations simultaneously, $h=4$, $k=3$, and $r=5$. Therefore the equation is $(x-4)^2+(y-3)^2=5^2$, which reduces at once to $x^2+y^2-8x-6y=0$.

We may also solve the problem by using form (14), which gives
$$1+49+D+7E+F=0,$$
$$64+36+8D+6E+F=0,$$
$$49+1+7D-E+F=0.$$

Solving, $\quad D=-8,\ E=-6,\ \text{and}\ F=0.$

A convenient graphical check on either solution is to draw the circle with compasses and observe whether or not it passes through the three given points.



48. Length of a Tangent. — Let t be the length of the tangent P_1T, let (h, k) be the coördinates of the center C, and let r be the radius. Then

$$t^2=CP_1^2-CT^2.$$

Using the distance formula, this becomes

$$t^2=(x_1-h)^2+(y_1-k)^2-r^2. \qquad (15)$$

It is observed that the expression for t is the form as the equation of the circle, with the coördi P_1 substituted for the variables.

Hence, *to find the length of a tangent from a given point to a given circle, transpose all terms of the given equation to the first member, substitute in the expression thus formed the coördinates of the given point, and extract the square root.*

Note the similarity of this to the formula for the distance from a line to a point.

Since the right-hand member of (15) is the same in form as the standard equation of the circle, it may also be written

$$t^2 = x_1^2 + y_1^2 + Dx_1 + Ey_1 + F. \tag{15a}$$

Example. — Find the length of the tangent from (5, 6) to the circle

$$x^2 + y^2 - 4x + 6y - 3 = 0.$$

Using formula (15*a*), we have

$$t^2 = 5^2 + 6^2 - 4 \cdot 5 + 6 \cdot 6 - 3 = 74.$$
$$t = \sqrt{74}.$$

PROBLEMS

1. Find the equation of the circle determined by the points:
(*a*) $(-1, 2)$, $(4, 2)$, $(-3, 4)$;
(*b*) $(0, 0)$, $(6, 0)$, $(0, -8)$;
(*c*) $(1, 1)$, $(1, 3)$, $(9, 2)$;
(*d*) $(-5, 0)$, $(0, 4)$, $(2, 4)$;
(*e*) $(2, 0)$, $(0, 0)$, $(-2, -2)$;
(*f*) $(6, 0)$, $(-2, 4)$, $(0, -6)$.

2. Find the equation of the circle which
(*a*) has the center $(3, 4)$ and passes through $(4, -3)$;
(*b*) has the center $(r, 0)$ and is tangent to the y-axis;
(*c*) passes through $(2, 0)$ and $(6, 0)$ and is tangent to the y-axis;
(*d*) has the line joining $(6, 2)$ and $(-8, 6)$ as a diameter;
(*e*) passes through $(3, 2)$ and $(-2, -1)$ with center on the x-axis;
(*f*) circumscribes the triangle whose sides are the lines $x = 5$, $y = -3$, $x - 2y = 7$;
(*g*) passes through $(9, 8)$ and is tangent to both axes.

Ans. (*b*) $y^2 = 2rx - x^2$;
(*e*) $5x^2 + 5y^2 - 8x = 41$;
(*g*) $x^2 + y^2 - 10x - 10y + 25 = 0$ or $x^2 + y^2 - 58x - 58y + 841 = 0$.

3. Find the equation of the circle which

(*a*) is tangent to the line $x + 2y - 10 = 0$ and passes through the points (3, 3) and (0, 0);

(*b*) has the center (1, 5) and is tangent to the line $3x - 4y + 7 = 0$;

(*c*) is tangent to the lines $2x - 3y = 0$ and $3x + 2y - 13 = 0$ and passes through the point $(-4, 6)$;

(*d*) is tangent to the line $3x - 4y = 2$ at the point (2, 1) and passes through the origin.

Ans. (*a*) $(x-1)^2 + (y-2)^2 = 5$,
or $(x - \frac{29}{9})^2 + (y + \frac{2}{9})^2 = \frac{845}{81}$;
(*c*) $(x+2)^2 + (y-3)^2 = 13$,
or $(x+22)^2 + (y-7)^2 = 325$;
(*d*) $2x^2 + 2y^2 + 7x - 24y = 0$.

4. Find the length of the tangent from the point (6, 8) to each circle of Problem 1, page 67.

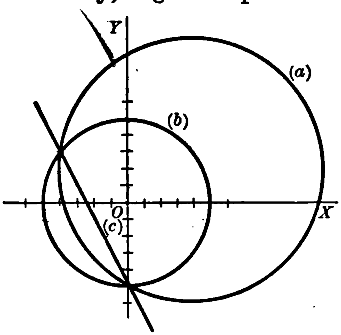
5. Find the least distance from the point (6, 8) to each circle of Problem 1, page 67. *Ans.* (*a*) $\sqrt{97} - 5$.

6. Find the coördinates of the points of contact of the tangents from the given point to the given circle:

(*a*) $x^2 + y^2 - 6x - 6y - 7 = 0$, $(-4, 4)$,
(*b*) $x^2 + y^2 = 17$, (5, 3).

Hint. — The construction of the tangents by the usual geometric process will suggest a method of solution.

49. Common Chord of Two Circles. — To find the common chord of two circles we may solve the equations simultaneously, to get the points of intersection, and then use the two-point form of the line equation. A better method will now be illustrated.



Let the two circles have the equations

$$x^2+y^2-8x-4y-44=0, \quad (a)$$

and

$$x^2+y^2-25=0. \quad (b)$$

Subtracting,

$$8x+4y+19=0. \quad (c)$$

By the theorem of § 41 the locus of (*c*) contains the points common to (*a*) and (*b*). Since it is of the first degree, it defines a straight line. Hence this line is the common chord.

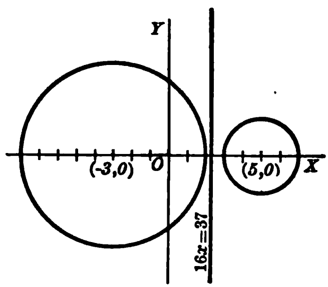
The equation of the common chord of two circles can always be obtained by subtracting the equation of one of them from that of the other, as above. For, since nothing above the first degree appears in the equation of a circle except the characteristic terms x^2+y^2, which are always present, these will cancel by subtraction, leaving a linear equation, whose locus contains the points of intersection of the two circles.

Exercise 2. Derive the general equation of the common chord of two circles. *Ans.* $(D-D')x+(E-E')y+(F-F')=0.$

Exercise 3. Prove that the common chord of two circles is perpendicular to their line of centers.

Hint. — The line of centers of two circles is determined by the points (h, k) and (h', k'). Hence its slope is $\dfrac{k-k'}{h-h'}$. Compare this with the slope of the common chord, found by solving the answer to Exercise 2 for y, and expressing the result in terms of h, k, h', and k'.

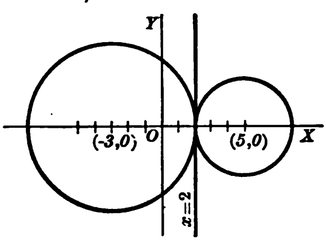
50. Radical Axis. — Consider the following two examples.



EXAMPLE 1. — Find the equation of the common chord of the circles

$$(x+3)^2+y^2=25$$
and $$(x-5)^2+y^2=4,$$
and the intersections of the circles.

Applying § 49, the equation is found to be $16x=37$; but the points of intersection are imaginary, as is seen by inspection of the equations, since the sum of the radii is 7, while the distance between the centers is $3+5=8$.



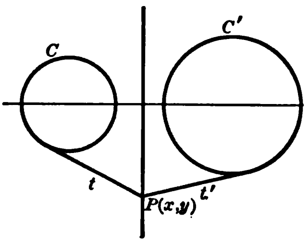
EXAMPLE 2. — Find the equation of the common chord of the circles

$$(x+3)^2+y^2=25$$
and $$(x-5)^2+y^2=9.$$

Here the distance between the centers is the same as the sum of the radii and the circles are tangent to each other at the point $(2, 0)$. Also we have by subtrac-

tion $x = 2$, whose locus, passing through the point (2, 0) and being perpendicular to the line of centers, is the common tangent.

Thus we see that the application of the rule for finding a common chord gives a common chord when the two circles intersect, a common tangent when they are tangent, and finally, when they do not meet, a line which has no apparent significance. We shall find, however, that the line in all three cases has a common property; viz. it is the locus of the point from which tangents to the two circles are equal.



To prove this, let t and t' be equal tangents from P to the circles C and C' respectively, of which the equations are

$$x^2 + y^2 + Dx + Ey + F = 0$$

and $$x^2 + y^2 + D'x + E'y + F' = 0.$$

From § 48, (15a), $t^2 = x^2 + y^2 + Dx + Ey + F.$

Also $t'^2 = x^2 + y^2 + D'x + E'y + F'.$

Hence $t^2 - t'^2 = (D - D')x + (E - E')y + F - F'.$

By hypothesis, $t = t'$, or $t^2 - t'^2 = 0.$

Therefore, $$(D - D')x + (E - E')y + F - F' = 0 \qquad (16)$$

is the equation of the locus of P.

But this is the equation obtained by eliminating the terms in x^2 and y^2. Hence we have the theorem:

The locus of the point from which the tangents to two circles are equal is the locus of the equation obtained by eliminating the terms in x^2 and y^2 between the equations of the circles.

This locus is called the *radical axis.* When the circles intersect, it is their common chord; and when they touch, it is their common tangent.

PROBLEMS

1. Find the equations of the radical axes and the points of intersection of the following pairs of circles :

(*a*) $x^2+y^2+4x+6y=7,$
$x^2+y^2-2x-6y+5=0;$

(*b*) $x^2+y^2-5x-5y=-11,$
$x^2+y^2-9=0;$

(*c*) $x^2+y^2+8x+18y=1,$
$3x^2+3y^2+20x+44y=11;$

(*d*) $x^2+y^2+4x+3y-32=0,$
$x^2+y^2+3x+5y-28=0;$

(*e*) $x^2+y^2-2x+4y-4=0,$
$2x^2+2y^2-4x+6y-7=0;$

(*f*) $x^2+y^2+7x-5y=8,$
$x^2+y^2+8x-4y=5.$

2. Find a point from which equal tangents may be drawn to each of the following circles :

(*a*) $x^2+y^2+3x+2y=6,$
$x^2+y^2-3x-2y=-2,$
$x^2+y^2-4x-8y=-12;$

(*b*) $x^2+y^2+4x-3y+5=0,$
$2x^2+2y^2+5x-6y+1=0,$
$x^2+y^2+5x-4y+5=0.$

3. Prove that the radical axes of any three circles taken in pairs meet in a point or are parallel.

4. In each pair of circles in Problem 1 find the line of centers, and prove that it is perpendicular to the radical axis.

51. Systems of Circles. — The geometric constants of the circle designated by h, k, and r determine the center and radius. If two of these constants are restricted by assigning conditions to them, numerical or otherwise, the equation will represent a *system of circles* defined more or less completely according to the nature of the conditions.

Thus, if $h = 2$ and $k = 4$, we have a system of concentric circles of varying radius. Again, if $h = k$ and $r = 10$, we have a system of equal circles whose centers lie on the line $x - y = 0$.

Systems having the Same Radical Axis. — If we use the theorem of § 41, we find that $C + k \cdot C' = 0$, where k is any constant and $C = 0$ and $C' = 0$ are the equations of two circles, will define a system of circles passing through the points of intersection of C and C'. But the radical axis of any two circles passes through the points of intersection, real or imaginary ; hence this equation defines a system of circles having the same radical axis.

The figure below illustrates the system of circles having the same radical axis as

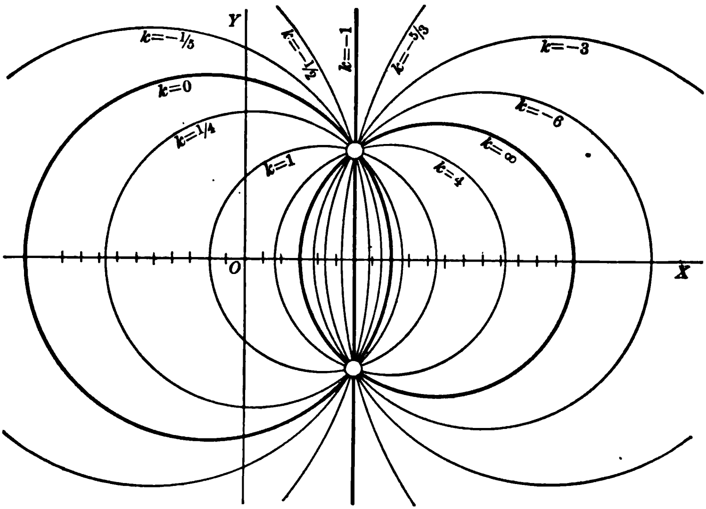
$$x^2 + y^2 + 4x - 96 = 0$$

and

$$x^2 + y^2 - 21x + 54 = 0.$$

These meet at the points (6, 6) and (6, − 6); the equation of the radical axis is $x = 6$. The equation of the system is

$$x^2 + y^2 + 4x - 96 + k(x^2 + y^2 - 21x + 54) = 0.$$



In the figure the circles drawn in heavy lines are the given circles and the others are the circles corresponding to

$$k = \tfrac{1}{4},\ 1,\ 4,\ -\tfrac{1}{5},\ -\tfrac{1}{2},\ -1,\ -\tfrac{5}{3},\ -3,\ -6.$$

Note that when k approaches -1, the circle becomes of infinite size and approaches as a limit the radical axis.

CHAPTER IV

THE PARABOLA

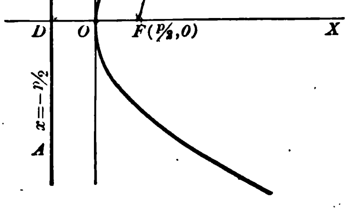
52. Conics. — *The locus of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed line is called a conic.*

The fixed point is called the *focus* and the fixed line the *directrix.* The line through the focus perpendicular to the directrix is called the *principal axis.* The constant ratio is called the *eccentricity* and is represented by the letter e.

There are three kinds of conics, classified as follows:

$$e = 1, \text{ the } \textit{parabola};$$
$$e < 1, \text{ the } \textit{ellipse};$$
$$e > 1, \text{ the } \textit{hyperbola}.$$

53. The Equation of the Parabola. — Let AB be the directrix, F the focus, and DF the principal axis of the parabola.



By the definition of the parabola, there is a point O, halfway between the focus and the directrix, which lies upon the locus. This is called the *vertex.* To derive the equation take the principal axis as the x-axis and the vertex as the origin. Call the distance between the directrix and the focus p. Then F has the coördinates $\left(\frac{p}{2}, 0\right)$, and the directrix has the equation $x = -\frac{p}{2}$.

Let $P(x, y)$ be any point on the locus. Draw NP perpendicular to AB, and join FP. By definition $FP = NP$,

that is, $$\sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} = x + \frac{p}{2}.$$

Squaring and reducing,

$$y^2 = 2\,px. \tag{17}$$

This is the standard equation of the parabola when the vertex is taken as the origin and the principal axis as the x-axis. If the principal axis is taken as the y-axis and the vertex as the origin, the variables are interchanged and the equation is

$$x^2 = 2\,py. \tag{17a}$$

If the focus is in a negative direction from the vertex, the right-hand members of these equations have minus signs.

The simplicity of these equations is due entirely to the choice of the coördinate axes. If the axes are chosen in any other way, a different and more complicated equation is obtained, but the curve itself is unaltered. Therefore we shall study the properties of the parabola and other conics by means of the simple forms and reserve the general equations for a later chapter.

Exercise 1. Derive equation (17*a*) from a figure.

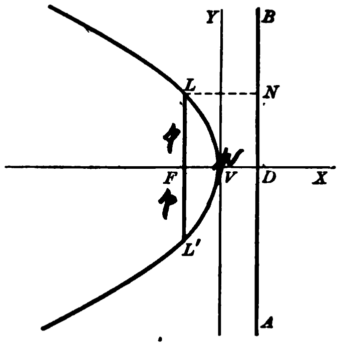
54. Discussion of the Equation.—The form of equation (17) shows that the parabola is symmetrical with respect to the principal axis, and that the parabola crosses this axis at the vertex only.

Since negative values of x make y imaginary, the curve lies wholly to the right of the vertex and has no point nearer the directrix. It recedes indefinitely from both axes, since $y = \pm\sqrt{2\,px}$ increases indefinitely with x. A different parabola is obtained for each value assigned to p.

Exercise 2. Derive the equation of the parabola, taking the focus at the left of the directrix, and discuss as in this section.

Ans. $y^2 = -2px.$

55. Latus Rectum. — The chord $L'L$ through the focus parallel to the directrix is called the *latus rectum.* By definition of the parabola, the distance of L from the focus is the same as that from the directrix.



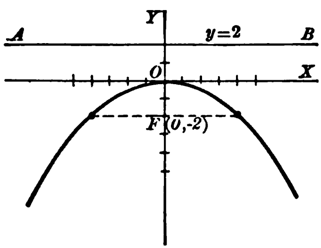
Hence

$$FL = LN = FD = p$$

and the length of the whole latus rectum is $2p$.

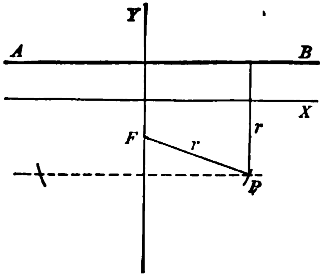
56. To Draw a Parabola. — Consider the equation $x^2 = -8y$. This is of the form $x^2 = -2py$, and hence is the equation of a parabola. To draw the curve we can make a table of values as usual, or proceed according to one of the two following methods.

Method 1. From the form of the equation, $2p = 8$, or $p = 4$. Since the sign before $2p$ is minus, the curve lies wholly below the x-axis.

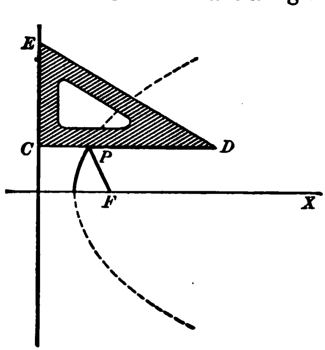


Hence the focus is $\left(0, -\frac{p}{2}\right)$, or $(0, -2)$, and the equation of the directrix is $y = 2$. Measuring off $p = 4$ to the right and left of the focus gives the ends of the latus rectum. These together with the vertex O make three points on the curve, which suffice for a sketch, as the general shape of a parabola is known.

Method 2. If a more accurate graph is desired, find the focus and directrix as above. Then with the focus as a center and a radius $r > \frac{p}{2}$ draw an arc. This will cut the line parallel to the directrix and r units from it in two points which lie on the parabola by definition. By using coördinate paper and varying r, as many points as are desired may be readily constructed.



57. Mechanical Construction of the Parabola. — Place a right triangle with one leg CE on the directrix. Fasten one end of a string whose length is CD at the focus F and the other end to the triangle at D. With a pencil at P keep the string taut. Then $FP = CP$; and as the triangle is moved along the directrix the point P will describe a parabola.



PROBLEMS

1. With coördinate paper and compasses plot a parabola having the latus rectum:

(*a*) 12; (*b*) 16.

2. Find the coördinates of the focus and the equation of the directrix of each of the following parabolas and draw the figure to scale:

(*a*) $y^2 = 8x$; (*c*) $2x^2 + 5y = 0$; (*e*) $y^2 = 6ax$;
(*b*) $y^2 = -8x$; (*d*) $y = 4x^2$; (*f*) $3x + 8y^2 = 0$.

3. What is the latus rectum of each parabola in Problem 2?

4. Write the equations of parabolas satisfying the following data:

(*a*) directrix $y = -8$, focus $(0, 8)$;
(*b*) directrix $x = 6$, focus $(-6, 0)$;
(*c*) directrix $y = 8$, vertex $(0, 0)$;
(*d*) latus rectum 12, vertex $(0, 0)$;
(*e*) vertex $(0, 0)$, one point $(3, -4)$.

Ans. (*a*) $x^2 = 32y$.

5. Derive the equation of the parabola when the principal axis is taken

(*a*) as the x-axis, with the origin at the focus;
(*b*) as the y-axis, with the origin at the focus;
(*c*) as the x-axis, with the origin on the directrix;
(*d*) as the y-axis, with the origin on the directrix.

Ans. (*a*) $y^2 = 2px + p^2$;
(*c*) $y^2 = 2px - p^2$.

6. Find the locus of the center of a circle passing through a given point and tangent to a given line.

7. Find the equation of the circle circumscribing the portion of the parabola $x^2 = 2py$ cut off by the latus rectum.

Ans. $2x^2 + 2y^2 - 5py = 0$.

8. Find the points of intersection of the parabolas $x^2 = 2py$ and $y^2 = 2px$.

9. In the parabola $y^2 = 2px$ an equilateral triangle is inscribed with one vertex at the origin. Find the length of a side.

Ans. $4p\sqrt{3}$.

10. Show that the distance of any point of the parabola $y^2 = 2px$ from the focus is $x + \frac{p}{2}$. (This is called the *focal radius* of the point.)

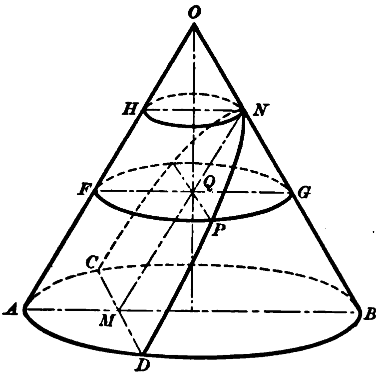
11. Show that the latus rectum of any parabola is a third proportional to the abscissa and ordinate of any point upon it.

12. Find the area of the right triangle inscribed in the parabola $y^2 = 2px$ which has a double ordinate as a hypothenuse and a vertex at the origin.

Ans. $4p^2$.

13. One end of a chord through the focus of a parabola which has the x-axis as the principal axis and the origin as the vertex is the point $(-8, 8)$. Find the coördinates of the other end.

58. The Parabola as a Conic Section. — In the right circular cone O–AB pass a plane through the vertex O and the diameter of the base AB. It will be perpendicular to the base and will contain the elements OA and OB, and also the axis and the center of every section parallel to the base.



Draw a plane perpendicular to the plane OAB intersecting it in the line MN parallel to OA, and cutting the surface of the cone in CND. Through P, any point of CND, and N pass planes parallel to the base, intersecting the plane OAB in HN and FG, which are the diameters of the circular sections thus formed.

Now QP, the intersection of the planes FPG and CND, is perpendicular to OAB and hence to the lines FG and NM. Taking N as the origin and NM as the x-axis, the coördinates of P are $x = NQ$ and $y = QP$.

Then
$$y^2 = \overline{QP}^2 = FQ \cdot QG.$$

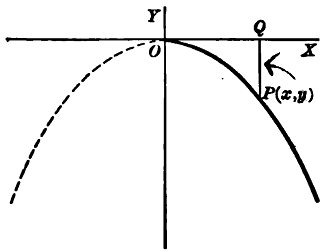
By elementary geometry,
$$FQ = HN, \text{ and } \frac{QG}{NQ} = \frac{HN}{OH}, \text{ or } QG = \frac{HN}{OH}x.$$

Substituting,
$$y^2 = \frac{\overline{HN}^2}{OH}x.$$

As HN and OH are constants, being independent of the position of P, $\frac{\overline{HN}^2}{OH}$ may be taken as $2p$. Thus the section CND is a parabola.

59. The Path of a Projectile. — A ball is thrown horizontally from a point above the ground. Find its path, neglecting air resistance.

Let v be the velocity of projection. Take the point of projection as the origin and a horizontal line through it as the x-axis.



Let $P(x, y)$ be the position of the ball after t seconds. If *v is measured in feet per second,* the ball will have moved horizontally a distance vt feet, that is,

$$x = OQ = vt.$$

But during the same time, by a principle of physics, the ball will have fallen a distance $\frac{1}{2}gt^2$ where $g = 32$ nearly. Hence

$$y = QP = -\tfrac{1}{2}gt^2.$$

Eliminating t between these equations,

$$x^2 = -\frac{2v^2}{g}y.$$

Therefore the locus is a parabola having its vertex at O and its latus rectum of length $\frac{2v^2}{g}$. The figure is drawn for $v = 64$ feet per second.

Later we shall find that the path of a projectile hurled in any direction is a parabola.

PROBLEMS

1. The altitude of a right circular cone is 10 inches and the diameter of the base is 8 inches. If the vertex of a parabola constructed as in § 58 bisects an element of the cone, find its latus rectum. *Ans.* $\frac{16}{29}\sqrt{29}$.

2. If two parabolic sections are made as in § 58, prove that their latus rectums are proportional to the distances of their vertices from the vertex of the cone.

3. If the vertex of a parabolic section is made to approach the vertex of its cone, what is the limiting form of the parabola? of its equation?

4. A bomb is dropped from an airship traveling east at 30 miles per hour at a height of 1 mile. Write the equation of its path, referred to the starting point as an origin. How far east will it be when it strikes the ground? *Ans.* 799 feet.

5. A ball was thrown horizontally on a level field at a height of 5 feet and struck the ground 100 feet from the starting point. What was its initial velocity?

6. An aviator flying horizontally at 45 miles per hour wishes to hit a target on the ground. He estimates his height above the ground at 1000 feet. How far away from the target should he be when he drops his projectile? *Ans.* 522 feet.

7. An airship which flies at 30 miles per hour has a device for releasing a projectile when it is directly over the center of the target. How large must the target be to insure hitting it when the height of the airship is

(*a*) $\frac{1}{2}$ mile; (*b*) 2 miles; (*c*) $1\frac{1}{2}$ miles?
Ans. (*a*) 1130 feet wide.

8. Sketch the parabolas satisfying the following data by finding the axis, vertex, and ends of the latus rectum:

(*a*) directrix $y = 5$, focus $(4, -3)$;
Ans. Axis $x = 4$, vertex $(4, 1)$, ends of latus rectum $(12, -3)$ and $(-4, -3)$.
(*b*) directrix $x = -5$, focus $(-1, 4)$;
(*c*) directrix $y = 6$, focus $(4, -2)$;
(*d*) directrix $x = 6$, focus $(-6, -6)$.

9. Derive the equations of the parabolas of Problem 8.
Ans. (*a*) $x^2 - 8x + 16y = 0$.

10. Show that the circle $x^2 - 2ax + y^2 = 0$ and the parabola $y^2 = 2px$ meet in but one point unless $a > p$.

11. Find the locus of the mid-points of the ordinates of the parabola $y^2 = 2px$.

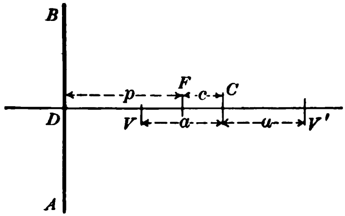
12. Find the locus of the ends of the latus rectum of the parabola $y^2 = 2px$ when p is allowed to vary.

CHAPTER V

THE ELLIPSE

60. Fundamental Constants. — In § 52 the ellipse was defined as a conic of eccentricity less than 1.

In the figure let AB be the directrix, F the focus, and DF the principal axis.



Since $e < 1$, there are two points V and V' on the principal axis, one on either side of the focus, satisfying the definition of the ellipse. These are called *vertices*, the point C midway between them is called the *center*, and the distances between V and C, and F and C designated by a and c respectively. Between the four constants a, c, p, and e exist certain important relations which we now proceed to derive.

Applying the definition of the ellipse to the points V and V', we have

$$FV' = eDV', \text{ or } a + c = e(c + p + a)$$

and

$$VF = eDV, \text{ or } a - c = e(c + p - a).$$

Adding and subtracting these equations,

$$a = e(c + p), \quad \text{or} \quad c + p = \frac{a}{e}, \tag{18}$$

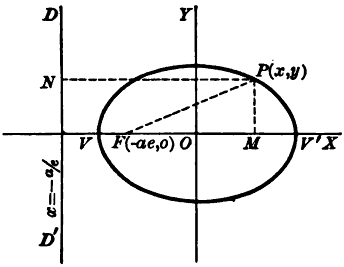
and

$$c = ae. \tag{19}$$

The first relation gives the distance between the center and the directrix in terms of a and e, and the second that between the center and the focus.

Exercise 1. Show how to find V and V' with ruler and compasses, when the directrix, focus, and eccentricity are given.

61. The Equation of the Ellipse. — To derive the equation of the ellipse take the principal axis for the x-axis and the center for the origin. Then F has the coördinates $(-ae, 0)$ and the directrix has the equation $x = -\frac{a}{e}$.



Let P be any point of the locus. Then $FP = e \cdot NP$. By the distance formula

$$FP = \sqrt{(x+ae)^2 + y^2},$$

and
$$NP = x + \frac{a}{e}.$$

Substituting these values,

$$\sqrt{(x+ae)^2 + y^2} = e\left(x + \frac{a}{e}\right) = ex + a.$$

Squaring and collecting terms,

$$(1 - e^2)x^2 + y^2 = a^2(1 - e^2).$$

Dividing by $a^2(1 - e^2)$, $\qquad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$

Since $e < 1$, the quantity $a^2(1 - e^2)$ is positive and $< a^2$. Calling this b^2, we have the standard form of the equation of the ellipse when the center is taken as the origin and the principal axis as the x-axis, namely:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{20}$$

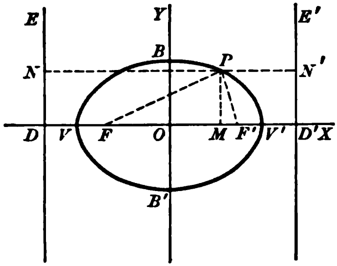
or
$$b^2x^2 + a^2y^2 = a^2b^2.$$

62. Discussion of the Equation. — *Intercepts and Extent.* — The intercepts of equation (20) are obviously $\pm a$ on the x-axis, and $\pm b$ on the y-axis.

Solving for y, we have $y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$, a form which shows that all values of x numerically greater than a must

be excluded. Similarly all values of y numerically greater than b must be excluded. Hence the ellipse lies wholly within the rectangle whose sides are $x = \pm a$, $y = \pm b$.

Axes of Symmetry. — The form of equation (20) shows that the ellipse is symmetrical with respect to both the x-axis and the y-axis, and hence is symmetrical with respect to the origin O.



The segment of the principal axis intercepted by the ellipse, namely VV', is called the *major* or *transverse axis.* The segment BB' of the y-axis is called the *minor* or *conjugate axis.* The major axis is of length $2a$ and the minor axis of length $2b$. O is called the center of the ellipse and is the mid-point of every chord of the ellipse which passes through it, since it is the center of symmetry.

From the definition of b (§ 61), we have at once

$$b^2 = a^2(1 - e^2) = a^2 - c^2,$$

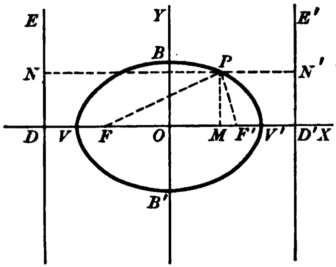
whence
$$a^2 = b^2 + c^2, \tag{21}$$

a relation showing that a is always greater than either b or c. As equations (18), (19), and (21) are mutually independent, all of the five constants of the ellipse can be found if any two of them are given.

Exercise 2. Find the distance from the focus to the ends of the minor axis.

63. Second Focus and Directrix. — On OV' take $OF' = OF$ and $OD' = OD$. Draw $D'E'$ parallel to DE. Then F' is also a focus and $D'E'$ the corresponding directrix of the ellipse. This is shown as follows:

Let P be any point on the ellipse. Draw $F'P$, and draw PN' perpendicular to $D'E'$. Calling $F'P = e \cdot PN'$, and substituting the corresponding values, the result is



$$\sqrt{(ae - x)^2 + y^2} = e\left(\frac{a}{e} - x\right).$$

Squaring both sides and collecting terms, this reduces to equation (20). Hence the ellipse of eccentricity e and having $D'E'$ for directrix and F' for focus is the same as that referred to DE and F.

Therefore the ellipse has two foci, $(\pm c, 0)$, and two directrices, $x = \pm \dfrac{a}{e}$.

PROBLEMS

1. Find the semi-major axis, the semi-minor axis, the eccentricity, the coördinates of the foci, and the equations of the directrices of the following ellipses: (a) $9x^2 + 25y^2 = 225$.

Solution. — Dividing by 225, we have

$$\frac{x^2}{25} + \frac{y^2}{9} = 1,$$

which is of the standard form. Hence $a = 5$, and $b = 3$. By (21) $c = 4$ and by (19) $e = \frac{4}{5}$. The foci are $(\pm 4, 0)$ and the directrices $x = \pm 6\frac{1}{4}$.

(b) $9x^2 + 36y^2 = 324$;
(c) $5x^2 + 8y^2 = 40$;
(d) $4x^2 + 9y^2 = 36$;
(e) $x^2 + 2y^2 = 2$;
(f) $4x^2 + 9y^2 = 1$;
(g) $9x^2 + 16y^2 = 16$.

2. Write the equation of the ellipse with the center at the origin and the principal axis as the x-axis, given:

(a) semi-major axis 5, eccentricity $\frac{3}{4}$;
(b) semi-minor axis 4, eccentricity $\frac{3}{5}$;
(c) vertex (13, 0), focus (5, 0);
(d) focus (4, 0), $e = \frac{1}{2}$;
(e) semi-major axis 6, directrix $x = 3\sqrt{5}$;
(f) directrix $x = -4$, eccentricity $\frac{4}{5}$;
(g) one focus $(\sqrt{2}, 0)$, one directrix $x = 2\sqrt{2}$.

Ans. (a) $7x^2 + 16y^2 = 175$;
(d) $3x^2 + 4y^2 = 192$;
(g) $x^2 + 2y^2 = 4$.

3. Express a in terms of e and p. *Ans.* $\dfrac{ep}{1-e^2}$.

4. Express c in terms of e and p.

5. Find the locus of a point moving so that the sum of its distances from the points $(\pm 12, 0)$ is 26.

6. Solve Problem 5 when the points are $(\pm c, 0)$ and the sum of the distances is $2a$.

7. From the circumference of a circle of radius 4 a series of perpendiculars is drawn to a diameter. A point moves so as to bisect each perpendicular. Find its locus.

8. Generalize Problem 7, calling the radius a and assuming that the point cuts off $\frac{1}{n}$th part of each perpendicular. *Ans.* $x^2 + n^2y^2 = a^2$.

9. Prove that the projection of a circle upon a plane is an ellipse.

Hint.—Let the plane upon which the circle is projected intersect the plane of the circle in a diameter, and let the angle between the two planes be α. Then a is the radius of the circle and $b = a \cos \alpha$.

10. Find the locus of the middle points of the ordinates of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

11. Lines are drawn from the center of an ellipse to its perimeter. Find the locus of a point which divides these lines in the ratio 2 : 3.
Ans. $25\,b^2x^2 + 25\,a^2y^2 = 4\,a^2b^2$.

12. Find the locus of the vertex of a triangle whose base is $2a$ and such that the product of the tangents of its base angles is $\dfrac{b^2}{a^2}$.

64. Focal Radii.—The distances of any point on the ellipse from the foci are called the focal radii. Denoting these by ρ and ρ', we have from §§ 61 and 63,

$$\rho = FP = e\left(\frac{a}{e} + x\right) = a + ex,$$

and

$$\rho' = F'P = e\left(\frac{a}{e} - x\right) = a - ex.$$

Adding these, $$\rho + \rho' = 2\,a.$$

Hence *the sum of the distances of any point on the ellipse from the foci is a constant and equal to the major axis.* This

in connection with Problem 6, page 89, shows that the ellipse may be defined as the locus of a point which moves so that the sum of its distances from two fixed points is a constant.

65. Latus Rectum. — The chord through either focus perpendicular to the major axis is called the latus rectum. If in the equation of the ellipse we put $x = \pm c$, then

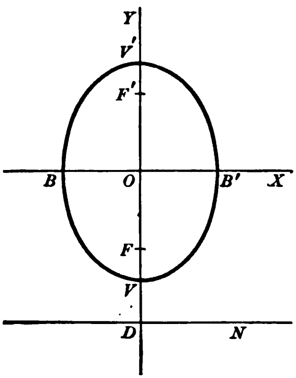
$$\frac{c^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{y^2}{b^2} = \frac{a^2 - c^2}{a^2} = \frac{b^2}{a^2}.$$

Therefore $\quad \dfrac{y}{b} = \pm \dfrac{b}{a}$, or $y = \pm \dfrac{b^2}{a}$. Hence the length of the latus rectum is $\dfrac{2\,b^2}{a}$.

As in the case of the parabola, the location of the ends of the latus rectum is a considerable aid to accuracy in sketching an ellipse of given dimensions.

Exercise 3. Show that the latus rectum is also given by $2\,ep$.

66. Ellipse with Foci on *Y*-axis. — If the foci are on the y-axis, this becomes the principal axis. Calling the major axis and minor axis of the ellipse $2\,a$ and $2\,b$ respectively as before, the effect on the equation of the ellipse is merely to interchange the variables x and y. It is then



$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1, \qquad (20\,a)$$

or $\qquad b^2y^2 + a^2x^2 = a^2b^2.$

In this form it is readily seen that the coördinates of the ver-

tices are $(0, \pm a)$; the coördinates of the foci are $(0, \pm c)$; the equations of the directrices are $y = \pm \frac{a}{e}$; and the lengths of the focal radii are $a \pm ey$.

Exercise 4. Prove as in § 61 that the central equation of an ellipse having the foci on the y-axis is $b^2y^2 + a^2x^2 = a^2b^2$.

Exercise 5. Prove that the focal radii of the ellipse with foci on the y-axis are $a \pm ey$.

67. The Circle as a Limiting Form. — If $b = a$, the equation of the ellipse becomes $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$, or $x^2 + y^2 = a^2$, which is a circle with the center at the origin and radius a. The constant c equals $\sqrt{a^2 - a^2} = 0$, and hence $e = \frac{c}{a} = 0$, and $\frac{a}{e}$ is infinite. The circle then may be considered as a limiting form of the ellipse with eccentricity zero, the two foci coinciding at the center and the directrices removed an infinite distance from the center.

68. Construction of an Ellipse by Points. — When a careful drawing of an ellipse is required, it is always possible to form the equation and compute a table of values. This, however, entails considerable labor, and it is usually better to proceed according to one of the following methods.

1. *Given the directrix, focus, and eccentricity.* Draw a parallel to the directrix at a distance k, and with a radius ek and the focus as a center draw an arc. Its intersections with the parallel are two points of the ellipse. By varying k, as many points as are desired may be obtained. When coördinate paper is used the parallels are already drawn; and if e is a convenient number, the points can be readily located.

2. *Given the foci and the major axis.* Lay off the major axis on coördinate paper. Then with a radius $\rho < 2a$ and

a focus as a center draw an arc. With the other focus as a center and $\rho' = 2a - \rho$ as a radius, draw another arc cutting the first in two points. These are points of the ellipse by § 64. By varying ρ we get as many points as are desired.

PROBLEMS

1. Find the vertices, foci, eccentricity, equations of the directrices, and length of the latus rectum of the ellipse whose equation is $25x^2 + 9y^2 = 225$.

Solution. — Dividing by 225, we have

$$\frac{x^2}{9} + \frac{y^2}{25} = 1.$$

Since a is always $> b$, $a = 5$ and $b = 3$, and the equation is of the form

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$$

Hence the vertices are $(0, \pm 5)$, the foci $(0, \pm 4)$, $e = \frac{4}{5}$, the directrices are $y = \pm 6\frac{1}{4}$, and the latus rectum $\frac{18}{5}$.

2. Investigate the following equations as in Problem 1.

(*a*) $36x^2 + 4y^2 = 144$;
(*b*) $3x^2 + 2y^2 = 1$;
(*c*) $5x^2 + 3y^2 = 1$;
(*d*) $9x^2 + 4y^2 = 4$;
(*e*) $16x^2 + 9y^2 = 144$;
(*f*) $25x^2 + 4y^2 = 100$.

3. Write the equation of the ellipse with the center at the origin, given:

(*a*) focus $(0, 6)$, directrix $y = 8$;
(*b*) vertex $(0, 4)$, focus $(0, 3)$;
(*c*) directrix $x = 6$, distance between foci 7;
(*d*) latus rectum $\frac{50}{13}$, vertex $(0, -13)$;
(*e*) one end of minor axis $(\sqrt{2}, 0)$, latus rectum 2.

Ans. (*a*) $4x^2 + y^2 = 48$; (*d*) $169x^2 + 25y^2 = 4225$.

4. Write the equations of the following ellipses with the center at the origin, the foci being first on the x-axis and second on the y-axis, unless the given data fix their position:

(*a*) $a = 8$, $b = 6$;
(*b*) $e = \frac{1}{2}$, $p = 3$;
(*c*) $c = 3$, $e = \frac{3}{5}$;
(*d*) $c = 5$, $b = 12$;
(*e*) vertices $(\pm 5, 0)$, one focus $(3, 0)$;
(*f*) foci $(\pm 5, 0)$, directrix $x = 15$;
(*g*) vertices $(\pm 13, 0)$, $e = \frac{5}{13}$.

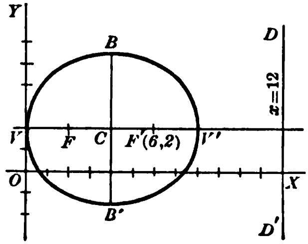
5. Construct by the methods of § 68 ellipses for which

(*a*) $p = 24,\ e = \frac{1}{3}$; (*c*) $a = 20,\ c = 12$;
(*b*) $p = 16,\ e = \frac{1}{2}$; (*d*) $a = 8,\ c = 5$.

6. Find the center, vertices, and ends of the minor axis of the following ellipses and sketch each curve.

(*a*) Focus (6, 2), directrix $x = 12$, $e = \frac{1}{2}$.

Solution. — Here $p = 6$. Using formulas (18) and (19), we have $a = 4$ and $c = 2$. Hence the center is (4, 2) and the vertices (8, 2) and (0, 2). By (21) $b = \sqrt{12}$ and the ends of the minor axis are $(4, 2 \pm \sqrt{12})$.



(*b*) Focus (4, 0), directrix $x = 0$, $e = \frac{2}{3}$;

(*c*) focus (1, 2), directrix $x = -4$, $e = \frac{1}{2}$;
(*d*) focus $(3, -4)$, directrix $y = 0$, $e = \frac{3}{4}$;
(*e*) focus $(-2, 3)$, directrix $y = -1$, $e = \frac{2}{3}$.

7. Derive the equations of the ellipses of Problem 6.

Ans. (*a*) $3x^2 - 24x + 4y^2 - 16y + 16 = 0$.

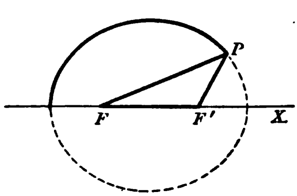
8. A line of constant length moves so that its extremities are on two lines at right angles. Find the locus of a point P on the line at a distance a from one end and b from the other.

9. Find the equation of an ellipse having its center at the origin and axes along the coördinate axes, which passes through:

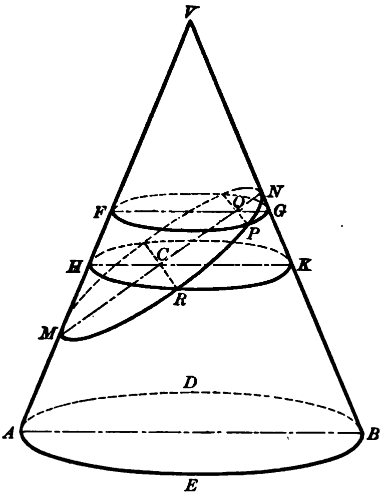
(*a*) (2, 4) and (5, −3); (*b*) (1, 2) and (2, 1).

10. Prove that the latus rectum of an ellipse is a third proportional to the major and minor axes.

69. Mechanical Construction of the Ellipse. — The relation of the focal radii may be used to construct the ellipse mechanically. On a drawing board fasten two tacks at the foci F and F', and tie a string about them of length equal to $2a + 2c$. If a pencil is placed in the loop FPF' and moved so as to keep the string taut, then $PF + PF'$ is a constant equal to $2a$, and P describes an ellipse (§ 64).



70. The Ellipse as a Section of a Cone. — Let $V\text{–}AEBD$ be a right circular cone, and VAB a plane determined by the vertex V and the diameter of the base AB. This plane will be perpendicular to the base and will contain the elements VA and VB, and also the axis of the cone and the center of every section parallel to the base.



Pass a plane perpendicular to the plane VAB, intersecting VA and VB at M and N. Through C, the mid-point, and Q, any other point of MN, pass planes parallel to the base, intersecting the plane VAB in FG and HK; these sections are circles and FG and HK are their diameters.

QP, the intersection of the planes FPG and MPN, is perpendicular to the plane VAB, and hence it is also perpendicular to the lines FG and MN. In the circles FPG and HRK,

$$\overline{QP}^2 = FQ \cdot QG \quad \text{and} \quad \overline{CR}^2 = HC \cdot CK.$$

The lines FQ and HC are parallel, hence the triangles FQM and HCM are similar.

$$\therefore \frac{FQ}{HC} = \frac{MQ}{MC} \text{ and } \frac{QG}{CK} = \frac{QN}{CN}.$$

Hence
$$\frac{\overline{QP}^2}{\overline{CR}^2} = \frac{MQ}{MC} \cdot \frac{QN}{CN}. \tag{a}$$

Let $CR = b$ and $MC = CN = a$, and we have

$$\frac{\overline{QP}^2}{b^2} = \frac{MQ \cdot QN}{a^2}. \qquad (b)$$

Now as Q moves along MN, P moves along the intersection of the conical surface and the cutting plane. Calling MN the x-axis and CR the y-axis, the coördinates of P will be $x = CQ$ and $y = QP$. Therefore $MQ = a + x$ and $QN = a - x$. Substituting in (b), we have

$$\frac{y^2}{b^2} = \frac{(a+x)(a-x)}{a^2},$$

which reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Thus the section of the cone is an ellipse.

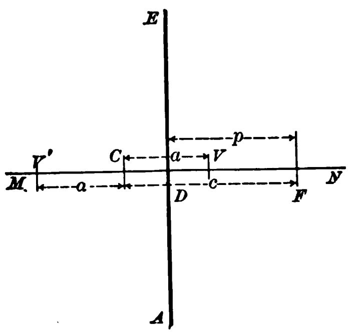
Exercise 6. Show that the section of a right circular cylinder made by a plane not parallel to the base is an ellipse.

CHAPTER VI

THE HYPERBOLA

71. Constants. — In § 52 the hyperbola was defined as a conic of eccentricity greater than 1.

In the figure let AE be the directrix, F the focus, and MN the principal axis. Let the distance between D and F be denoted by p, and the eccentricity by e.



Since $e > 1$, there are two points of the hyperbola, V and V', on the principal axis, one on each side of the directrix. As in the ellipse these are called *vertices*; the point C midway between them is called the *center*; and the distances between V and C, and F and C are designated by a and c respectively.

Applying the definition of the hyperbola to the points V and V', we have

$$VF = e \cdot DV \text{ or } c - a = e[a - (c - p)],$$

and $$V'F = e \cdot V'D \text{ or } c + a = e[a + (c - p)].$$

Subtracting and adding these equations,

$$a = e(c - p) \text{ or } c - p = \frac{a}{e} \tag{22}$$

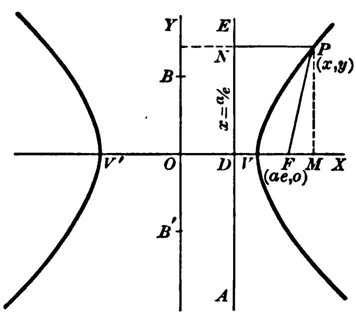
and $$c = ae. \tag{23}$$

The first relation gives the distance between the center and directrix in terms of a and e, and the second that between the center and focus.

Exercise 1. From the definition of the hyperbola find DV and $V'D$ in terms of p and e. *Ans.* $\dfrac{p}{e \pm 1}$.

Exercise 2. Using the results of Exercise 1, show that C is to the left of the directrix.

72. The Equation of the Hyperbola. — To derive the equation of the hyperbola take the principal axis as the x-axis and the center as the origin. Then F has the coördinates $(ae, 0)$, and the directrix has the equation $x = \dfrac{a}{e}$.



Let P be any point on the locus. Draw PN perpendicular to AE and join FP. Then

$$FP = e \cdot NP.$$

But $\quad FP = \sqrt{(x - ae)^2 + y^2}$, and $NP = x - \dfrac{a}{e}$.

Hence $$\sqrt{(x-ae)^2 + y^2} = e\left(x - \frac{a}{e}\right).$$

Squaring and collecting terms,

$$(1 - e^2)x^2 + y^2 = a^2(1 - e^2),$$

or $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.$$

In this form the equation of the hyperbola appears identical with that of the ellipse (§ 61). However, $e > 1$ and $a^2(1 - e^2)$ is negative. Calling $b^2 = a^2(e^2 - 1)$, we have the hyperbola equation in its usual form, when the center is

taken as the origin and the principal axis as the x-axis, namely,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{24}$$

or

$$b^2x^2 - a^2y^2 = a^2b^2.$$

73. Discussion of the Equation. — *Intercepts and Extent.* — Solving equation (24) for x, we have $x = \pm \frac{a}{b}\sqrt{y^2 + b^2}$, which shows that the x-intercepts are $\pm a$, and that no value of y needs to be excluded.

Solving for y, $y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$, showing that there are no y-intercepts and also that y is imaginary for all values of x between $\pm a$, but if x is numerically greater than a, y is real and increases indefinitely as $|x|$ increases. Thus the hyperbola consists of two infinite branches, lying without the lines $x = \pm a$.

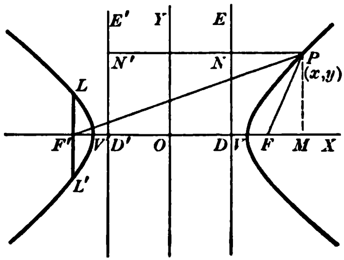
Symmetry.— The form of (24) shows that the hyperbola is symmetrical to the x- and y-axes and hence to the origin. The segment $V'V$ of the principal axis is called the *transverse axis* and is of length $2a$. The segment $B'B$ of the y-axis, where $B'O = OB = b$, is called the *conjugate axis*. From the definition of b in § 72 we have at once

$$c^2 = a^2 + b^2, \tag{25}$$

a form showing that for the hyperbola c is greater than either a or b, while there is no restriction on the relative sizes of a and b. It is for this reason that the terms major and minor axis are not used.

74. Second Focus and Directrix. — As in the case of the ellipse there is a second focus and a second directrix. For on the principal axis take $OF' = OF$ and $OD' = OD$, and

erect the perpendicular $D'E'$. Then the same curve will be generated by using the focus F', the directrix $D'E'$, and the same value for e.



For we have $F'P = eN'P$; but $F'P = \sqrt{(ae+x)^2+y^2}$, and $N'P = x + \frac{a}{e}$. Therefore

$$(ae+x)^2 + y^2 = (ex+a)^2,$$

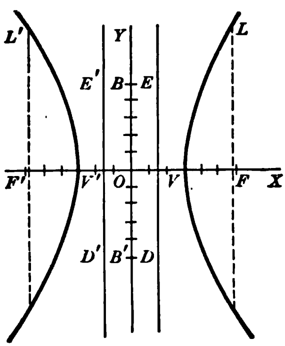
and this reduces to (24).

Hence the hyperbola has two foci, $(\pm c, 0)$, and two directrices, $x = \pm \frac{a}{e}$.

75. Latus Rectum. — The chord $L'L$ through either focus perpendicular to the principal axis is called the latus rectum and it is seen (as in the case of the ellipse) that its length is $\frac{2b^2}{a}$.

Exercise 3. Show that the length of the latus rectum is also $2\,ep$.

PROBLEMS



1. Find the semi-transverse and semi-conjugate axes, the coördinates of the foci, the eccentricity, and the equations of the directrices of the hyperbola having the equation $25x^2 - 9y^2 = 225$.

Solution. — Dividing by 225, we have the standard form

$$\frac{x^2}{9} - \frac{y^2}{25} = 1.$$

Hence $a = 3$, $b = 5$, $c = \sqrt{34}$, and $e = \frac{1}{3}\sqrt{34}$. The foci are $(\pm\sqrt{34}, 0)$, and the directrices are $x = \pm\frac{9}{\sqrt{34}}$. As the vertices $(\pm 3, 0)$ give only two points, we lay off the semi-latus rectum $\frac{b^2}{a} = \frac{25}{3}$ from the foci, getting four more points.

2. Find the properties of the following hyperbolas as in Problem 1:

(a) $9x^2 - 16y^2 = 144$; (d) $x^2 - 2y^2 = 1$;
(b) $25x^2 - 9y^2 = 225$; (e) $3x^2 - 5y^2 = 15$;
(c) $x^2 - 9y^2 = 9$; (f) $25x^2 - 15y^2 = 144$.

3. Find the equation of the hyperbola having its center at the origin and its foci on the x-axis, if

(a) $a = 4$, $b = \frac{5}{2}$; (d) $a = 5$, $e = \frac{4}{3}$;
(b) $b = 3$, $c = 5$; (e) $c = 5$, $a = 2\sqrt{6}$;
(c) $b = 4$, $e = 2$; (f) $c = \sqrt{5}$, $e = \frac{\sqrt{5}}{2}$.

4. Express a in terms of e and p. *Ans.* $\frac{ep}{e^2 - 1}$.

5. Express c in terms of e and p.

6. Find the locus of a point moving so that the difference of its distances from the points $(\pm 13, 0)$ is 24.
Ans. $25x^2 - 144y^2 = 3600$.

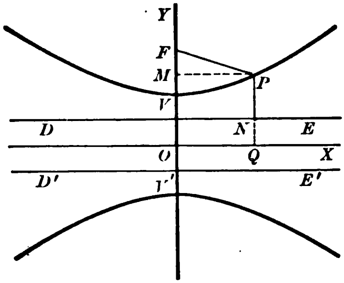
7. Solve Problem 6 when the points are $(\pm c, 0)$ and the difference of the distances is $2a$.

8. Define the hyperbola from the conditions of Problem 7.

9. Find the equation of a hyperbola having its center at the origin and its foci on the x-axis and passing through:
(a) $(3, 2)$ and $(-\sqrt{3}, 1)$; (b) $(3, 0)$ and $(5, -4)$.

10. Find the equation of a hyperbola whose vertex is midway between the center and the focus. *Ans.* $3x^2 - y^2 = 3a^2$.

76. Hyperbola with Foci on the Y-Axis. — If we describe a hyperbola with the foci on the y-axis, with its transverse axis $2a$, distance between the foci $2c$, and eccentricity e, the effect on the original equation will be to interchange the variables x and y, and the equation will become



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \qquad (24a)$$

or $b^2y^2 - a^2x^2 = a^2b^2$.

The vertices of this hy-

perbola are $(0, \pm a)$, the foci $(0, \pm c)$, and the directrices

$$y = \pm \frac{a}{e}.$$

Exercise 4. Derive equation (24*a*) as in § 72.

77. Focal Radii. — As in the ellipse the distances of any point on the hyperbola from the foci are called the focal radii and designated by ρ and ρ'. From the figure of § 74 we have

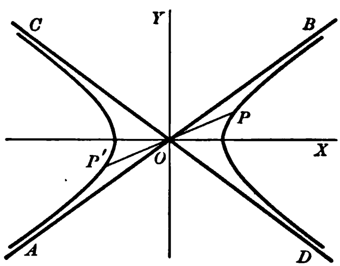
$$\rho = FP = e \cdot NP = e\left(x - \frac{a}{e}\right) = ex - a,$$

and

$$\rho' = F'P = e \cdot N'P = e\left(x + \frac{a}{e}\right) = ex + a.$$

Subtracting these, we have $\rho' - \rho = 2a$. Hence *the difference of the distances of any point of the hyperbola from the foci is constant and equal to the transverse axis.* This agrees with the results of Problems 7 and 8, page 100, which show that the hyperbola may be defined as the locus of a point moving so that the difference of its distances from two fixed points is constant.

78. Asymptotes. — If a line is drawn through the center of a hyperbola intersecting it at P, and P is made to move off to infinity along the curve, the line will turn about O and approach one of two limiting lines. We call these asymptotes, defining them as lines through the center of the hyperbola along which the curve recedes to infinity.



The equation of any line PP' through the center or origin is $y = mx$. The coördinates of its intersections with the hyperbola are given by solving this simultaneously with

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$ Substituting, we have $\frac{x^2}{a^2}-\frac{m^2x^2}{b^2}=1$, which gives

$$x = \pm \frac{ab}{\sqrt{b^2-a^2m^2}}.$$

Now if P is to move off to infinity, x must become infinite, *i.e.* the denominator must become zero. This gives

$$b^2 - a^2m^2 = 0 \quad \text{or} \quad m = \pm\frac{b}{a}.$$

Hence the slopes of the asymptotes are $\pm\frac{b}{a}$ and their equations

$$y = \pm\frac{b}{a}x, \tag{26}$$

or

$$bx \pm ay = 0.$$

These equations may be combined by § 42 in the form

$$b^2x^2 - a^2y^2 = 0, \quad \text{or} \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=0. \tag{26a}$$

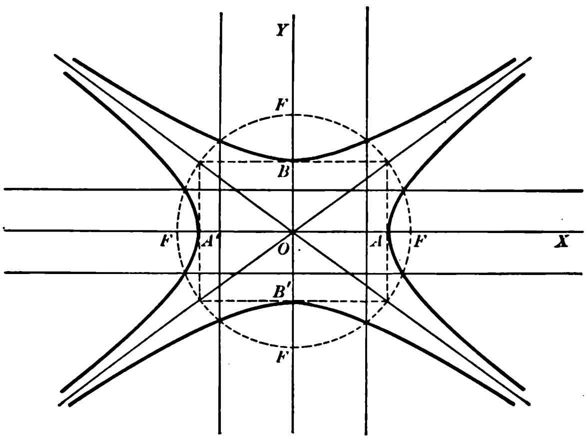
In case the vertices are on the y-axis, x and y are interchanged, giving $\frac{y^2}{a^2}-\frac{x^2}{b^2}=0$. In either case, *to find the equations of the asymptotes, write the equation of the hyperbola in standard form, replace the 1 by 0, and factor.*

79. Conjugate Hyperbolas. — Since the relative sizes of a and b are immaterial, a hyperbola may be described having the segment $2b$ for a transverse axis and the segment $2a$ for a conjugate axis. The hyperbola with transverse axis $A'A$ $(=2a)$ along the x-axis and conjugate axis $B'B$ $(=2b)$ along the y-axis has the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The hyperbola with transverse axis $B'B$ $(=2b)$ along the y-axis and

conjugate axis $A'A$ $(= 2\,a)$ along the x-axis has the equation $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$. The two hyperbolas

$$(a)\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad (b)\quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

are called *conjugate hyperbolas.* The transverse axis of each is the conjugate axis of the other.



Since $c^2 = a^2 + b^2$, the foci of both hyperbolas are at the same distance from the center. Those of (a) are $(\pm\, c,\, 0)$; those of (b) are $(0,\, \pm\, c)$. If we let e_1 and e_2 be the respective eccentricities, we have $e_1 = \frac{c}{a}$, $e_2 = \frac{c}{b}$, whence $e_1 : e_2 = b : a$.

The directrices of (a) are $x = \pm\, \frac{a}{e_1}$, of (b) $y = \pm\, \frac{b}{e_2}$. From the equations it is evident that conjugate hyper the same asymptotes.

This property is useful in sketching a pair of hyperbolas. Draw the rectangle of the given ax a rectangle having its sides of length $2\,a$ and

metrical with respect to the axes, and draw the diagonals. As these have slopes $\pm \frac{b}{a}$, they are the asymptotes. The circle circumscribing the rectangle has the radius $\sqrt{a^2+b^2}=c$, and therefore intersects the axes at the foci of the required hyperbolas. With the rectangle and asymptotes as guiding lines the curves can be sketched accurately.

Exercise 5. Prove that the directrices of a hyperbola cut the conjugate hyperbola at points on the circle drawn through the four foci.

80. Equilateral Hyperbolas. — When $a = b$, the equation of the hyperbola reduces to

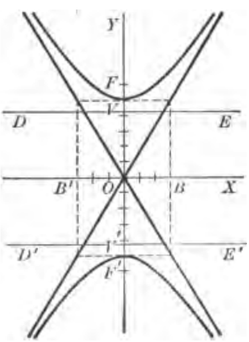
$$x^2 - y^2 = a^2$$

and that of the conjugate to

$$y^2 - x^2 = a^2.$$

The asymptotes are inclined to the axes at 45° and are perpendicular to each other. Such hyperbolas are called *equilateral* or *rectangular* hyperbolas.

81. Construction of a Hyperbola by Points. — A hyperbola may be plotted accurately by forming a table of values from the equation or by proceeding according to methods similar to those used for the ellipse (§ 68).



PROBLEMS

1. Find the semi-transverse and semi-conjugate axes, the coördinates of the foci, the eccentricity, and the equations of the directrices of the hyperbola whose equation is $25x^2 - 9y^2 + 225 = 0$.

Solution. — Transposing and dividing by -225, we have

$$-\frac{x^2}{9}+\frac{y^2}{25}=1.$$

Since the term in y is positive, this is of the form

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1.$$

Hence $a=5$, $b=3$, $c=\sqrt{34}$, and $e=\frac{1}{5}\sqrt{34}$.

The foci are $(0, \pm\sqrt{34})$ and the directrices are $y = \pm\dfrac{25}{\sqrt{34}}$. Drawing the asymptotes, the curve is readily sketched.

2. Find the properties of the following hyperbolas in the same manner as in Problem 1:

(*a*) $9x^2 - y^2 = 18$; (*d*) $3x^2 - 4y^2 - 12 = 0$;
(*b*) $5y^2 - 10x^2 = 50$; (*e*) $9x^2 - 18y^2 + 16 = 0$;
(*c*) $x^2 - 9y^2 = 9$; (*f*) $16x^2 - 25y^2 = 64$.

3. Write the equations of the hyperbolas conjugate to those of Problem 2 and of their common asymptotes.

4. Find the equations of the hyperbolas of Problem 3, page 100, if the foci are taken on the y-axis. *Ans.* (*a*) $64x^2 - 25y^2 + 400 = 0$.

5. Construct by points hyperbolas for which

(*a*) $p = 24,\ e = 3$; (*c*) $a = 20,\ c = 32$;
(*b*) $p = 16,\ e = 2$; (*d*) $a = 5,\ c = 8$.

6. Find the center and lengths of the semi-axes of each of the following hyperbolas. Draw the asymptotes and sketch each curve.

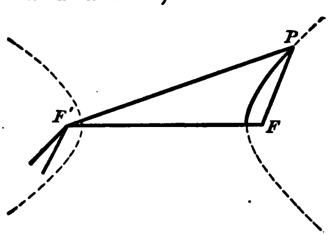
(*a*) Focus $(6, 2)$, directrix $x = 12$, $e = 2$;
Ans. $c = 8$, $C = (14, 2)$, $a = 4$, $b = 4\sqrt{3}$.
(*b*) focus $(4, 0)$, directrix $x = 0$, $e = \frac{3}{2}$;
(*c*) focus $(1, 2)$, directrix $x = -4$, $e = 2$;
(*d*) focus $(3, -4)$, directrix $y = 0$, $e = \frac{4}{3}$;
(*e*) focus $(-2, 3)$, directrix $y = -1$, $e = \frac{3}{2}$.

7. Derive the equations of the hyperbolas of Problem 6.
Ans. (*a*) $3x^2 - 84x - y^2 + 4y + 536 = 0$.

8. Show that the distance of any point on an equilateral hyperbola from the center is a mean proportional between the focal radii.

9. Show that for a pair of conjugate hyperbolas $e_1^2 + e_2^2 = e_1^2 e_2^2$.

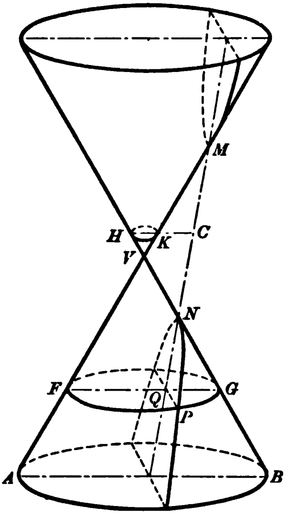
82. Mechanical Construction of the Hyperbola. — Place thumb tacks in a drawing board at F and F', the foci. Let a string be tied to a pencil at P and looped about the tacks as in the figure. If the ends are drawn in together, P will describe a hyperbola since



$$(F'F + FP) - F'P$$

remains constant. For $F'F$ is constant; hence $F'P - FP$ is a constant.

83. The Hyperbola as a Conic Section. — Let the cutting plane intersect the given cone so as to meet some of the elements produced. Follow the construction of § 70 and choose the axes in the same manner, viz., origin at C, and x-axis MN. Then



$$y^2 = \overline{QP}^2 = FQ \cdot QG.$$

Also

$$\frac{FQ}{KC} = \frac{MQ}{MC} \quad \text{and} \quad \frac{QG}{HC} = \frac{NQ}{CN}.$$

Substituting,

$$y^2 = \frac{MQ \cdot NQ}{MC \cdot CN}(KC \cdot HC);$$

or if

$$MC = CN = a \text{ and } KC \cdot HC = b^2,$$

$$y^2 = \frac{b^2}{a^2}(MQ \cdot NQ).$$

But $$MQ = x + a, \quad NQ = x - a,$$

whence $$y^2 = \frac{b^2}{a^2}(x^2 - a^2) \quad \text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Hence the plane cuts the surface of the cone in one branch of a hyperbola and the conical surface formed by prolonging the elements through the vertex in the other branch.

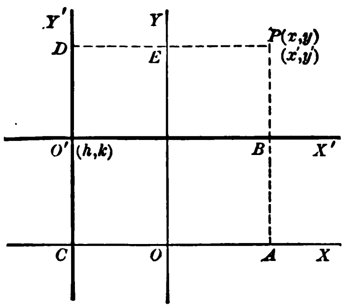
CHAPTER VII

TRANSFORMATION OF COÖRDINATES AND SIMPLIFICATION OF EQUATIONS

84. Change of Axes. — The position of the axes of coördinates to which a given locus is referred is arbitrary. It may be changed at will and the equation of the locus altered to correspond by substituting for the former coördinates their values in terms of coördinates measured from the new axes.

The advantage to be secured by a change of axes is usually a simplification of the equation, and this is best secured by choosing a position of symmetry when possible.

When the new axes are drawn parallel to the old, the transformation of the equation is called a *transformation by translation.* When they are drawn through the same origin oblique to the axes but still perpendicular to each other, the transformation is called a *transformation by rotation.*



85. Formulas of Translation. — In the figure let the original set of axes be OX and OY; the new axes $O'X'$ and $O'Y'$ with the origin at O', which has the coördinates (h, k) with reference to OX and OY. Let P be any point in the plane which has coördinates (x, y) with reference to the old axes, and (x', y') with reference to the new ones. Then

$$x = EP, \qquad x' = DP, \qquad h = OC = ED,$$
$$y = AP, \qquad y' = BP, \qquad k = CO' = AB.$$

From the figure

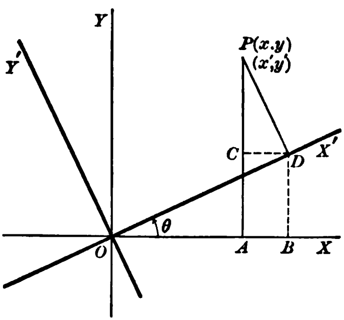
$$EP = DP - DE = DP + ED \text{ and } AP = BP + AB.$$

Hence $$x = x' + h,$$
and $$y = y' + k. \tag{27}$$

These are the *formulas of translation.*

By using these substitutions a given equation is transformed into a new equation in x' and y', which is the equation of the locus of the given equation with respect to coördinate axes drawn through the point (h, k) parallel to the old axes.

86. Formulas of Rotation. — Let the original axes be OX and OY, the new axes OX' and OY', and the angle of rotation θ. P is any point in the plane with coördinates (x, y) and (x', y') with reference to the old and new axes respectively. Then



$$x = OA, \qquad x' = OD,$$
$$y = AP, \qquad y' = DP.$$

Since the sides of angle CPD are perpendicular to those of θ, $CPD = \theta$. Hence we have

$$CD = y' \sin\theta, \quad CP = y' \cos\theta, \quad OB = x' \cos\theta, \quad BD = x' \sin\theta.$$

But $x = OA = OB - CD$ and $y = AP = BD + CP$.

Substituting, we get

$$x = x' \cos\theta - y' \sin\theta,$$
$$y = x' \sin\theta + y' \cos\theta, \tag{28}$$

which are the *formulas of rotation.*

These substitutions transform a given equation into a new equation with variables x' and y', in which the coördinate axes are drawn through the old origin but inclined at an angle θ to the old axes.

PROBLEMS

General Directions. — Usually the equation obtained by transformation is a known form and its locus can be identified. When this is not the case, the table of values should be computed from the new equation and the curve plotted on the new axes, since they are axes of symmetry. Frequently the intercepts with respect to both old and new axes, together with considerations of symmetry, make an extended table of values unnecessary.

1. Translate the origin to the point indicated and transform the equation to correspond. Draw the curve and both sets of axes.

(a) $4x^2 - y^2 - 8x + 4y + 4 = 0$, $(1, 2)$.

Ans. $4x'^2 - y'^2 + 4 = 0$.

(b) $2x^2 + y - 5 = 0$, $(0, 5)$;
(c) $y^2 - 8x = -24$, $(3, 0)$;
(d) $x^2 + y^2 = 6x - 4y$, $(3, -2)$;
(e) $y = x^3 + 6x^2 + 8x + 3$, $(-2, 3)$;
(f) $y = b + (x - a)^3$, (a, b);
(g) $16x^2 - 64x + 9y^2 + 18y - 71 = 0$, $(2, -1)$.

2. Rotate the axes through the angle indicated and transform the equation to correspond. Draw the curve and both sets of axes. Identify known loci.

(a) $xy = 8$, $\theta = \frac{\pi}{4}$;

(b) $x^2 - y^2 = 16$, $\theta = \frac{\pi}{4}$;

(c) $xy = -8$, $\theta = \frac{\pi}{4}$;

(d) $x^2 + y^2 - 4xy + 9 = 0$, $\theta = \frac{\pi}{4}$;

(e) $x^2 + 4xy + y^2 = 16$, $\theta = \frac{\pi}{4}$;

(f) $29x^2 - 24xy + 36y^2 = 180$, $\theta = \sin^{-1}\frac{3}{5}$;
(g) $17x^2 - 312xy + 108y^2 = 900$, $\theta = \sin^{-1}\frac{3}{5}$.

Ans. (a) $x'^2 - y'^2 = 16$; (f) $4x'^2 + 9y'^2 = 36$.

3. Prove that $x^2 + y^2 = r^2$ is unchanged by rotating the axes through any angle.

4. Find the coördinates of $P(4, 3)$ and $P(2, -1)$ when the origin is translated to

(*a*) $(6, 5)$; (*b*) $(4, -2)$; (*c*) $(-2, -1)$, (*d*) $(-4, -5)$.

Ans. (*a*) $(-2, -2)$, $(-4, -6)$.

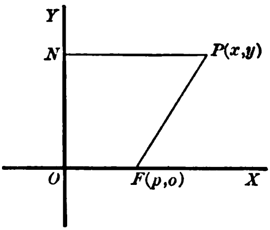
5. Find the coördinates of

(*a*) $(3, 5)$; (*b*) $(-2, 2)$; (*c*) $(4, -2)$; (*d*) $(-6, 6)$

when the axes are rotated 45°; $-45°$; 90°.

Ans. (*a*) $(4\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, 4\sqrt{2})$, $(5, -3)$.

87. Application to the Conics. — Let us derive the equation of the conic, taking the directrix as the y-axis and the principal axis as the x-axis. We have by definition



$$FP = eNP,$$

whence $\sqrt{(x-p)^2 + y^2} = ex$,

or $$(x-p)^2 + y^2 = e^2x^2. \quad (29)$$

This is a general form of the conic equation, true for all values of e. From it by translation of axes we can derive the forms previously obtained.

For the parabola $e = 1$. Then (29) reduces to

$$y^2 = 2\,px - p^2.$$

As the vertex is halfway between the directrix and the focus, its coördinates are $\left(\frac{p}{2}, 0\right)$. Thus to move the origin to the vertex we substitute $x = x' + \frac{p}{2}$, $y = y'$.

Simplifying and dropping primes, we get

$$y^2 = 2\,px,$$

the standard equation of the parabola.

For the ellipse we have seen in Chapter V that the distance from the directrix to the center is $p + c$. Therefore

the coördinates of the new origin are $(p + c, 0)$. Substituting $x = x' + p + c$, $y = y'$ in (29), we get

$$(x' + c)^2 + y'^2 = e^2(x' + p + c)^2.$$

But $c = ae$, $p + c = \frac{a}{e}$. Using these relations and dropping primes, we obtain the equation,

$$(x + ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2,$$

which can easily be reduced to

$$(1 - e^2)x^2 + y^2 = a^2(1 - e^2).$$

Dividing by the right-hand member, and putting $a^2(1-e^2)=b^2$, we have the standard form

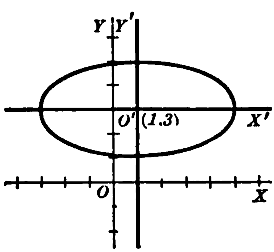
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The central equation of the hyperbola can be derived in the same way. The student should work out the reduction for himself, bearing in mind that for the hyperbola the distance *from* the directrix *to* the center is $p - c = -\frac{a}{e}$, and that $b^2 = a^2(e^2 - 1)$.

88. Test for Axes of Symmetry. — To find axes of symmetry parallel to the coördinate axes, proceed as follows: Solve the equation for y in terms of x. If the solution is of the form $y = k \pm f(x)$, the line $y = k$ is an axis of symmetry. This is due to the fact that each value of x, as $x = a$, gives two values of y, and thus two points, one $f(a)$ above the line $y = k$, the other $f(a)$ below. Similarly, if the solution for x is of the form $x = h \pm g(y)$, the line $x = h$ is an axis of symmetry. Since the lines $x = h$ and $y = k$ are perpendicular to each other, a curve symmetrical with respect to both of these lines has their intersection (h, k) as a center of symmetry. (See § 22.)

89. Simplification of Equations by Translation. — In case the curve has two axes of symmetry, they can be found by the method just given and the translation made by formula. When this is not true, the second method illustrated in the following examples must be used.

EXAMPLE 1. — Simplify $x^2 + 4y^2 - 2x - 24y + 21 = 0$.



Solution. — First Method. Solving for x and y, we have

$$x = 1 \pm f(y)^*$$

and $$y = 3 \pm g(x).$$

Hence the axes of symmetry are

$$x = 1 \text{ and } y = 3$$

and the center of symmetry is (1, 3).

Substitute

$$x = x' + 1 \text{ and } y = y' + 3$$

in the original equation, and we have

$$(x' + 1)^2 + 4(y' + 3)^2 - 2(x' + 1) - 24(y' + 3) + 21 = 0.$$

$$x'^2 + 4y'^2 = 16.$$

$$\frac{x'^2}{16} + \frac{y'^2}{4} = 1.$$

Thus the curve is an ellipse with its center at (1, 3) and semi-axes 4 and 2.

Second Method. In the given equation substitute $x = x' + h$ and $y = y' + k$. It becomes, after collecting coefficients,

$$x'^2 + (2h - 2)x' + 4y'^2 + (8k - 24)y' + (h^2 + 4k^2 - 2h - 24k + 21) = 0.$$

If the curve is symmetrical with respect to the new axes, there can be no terms of the first degree in the new equation. This is the case if

$$2h - 2 = 0 \text{ and } 8k - 24 = 0, \text{ or } h = 1,\ k = 3.$$

Hence the center of symmetry is (1, 3) and the transformed equation becomes $x'^2 + 4y'^2 - 16 = 0$ as before, on substituting $h = 1$, $k = 3$.

* In solving for x, the terms not involving x may be disregarded, as we are interested only in showing that the solution is of the form

$$x = h \pm f(y).$$

EXAMPLE 2. — Simplify $y^2 + 8x + 4y - 20 = 0$.

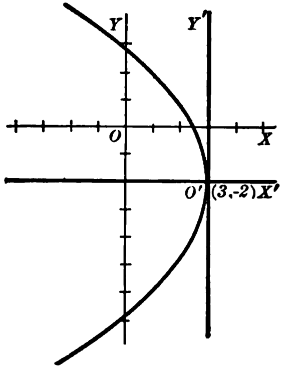
Solution. — If the equation is solved for x no axis of symmetry is revealed, and so we must use the second method. Put $x = x' + h$, and $y = y' + k$. Collecting coefficients, the equation becomes

$$y'^2 + 8x' + (2k + 4)y' + (k^2 + 8h + 4k - 20) = 0.$$

Evidently the terms in y'^2 and x' cannot be eliminated. But if $2k + 4 = 0$ and $k^2 + 8h + 4k - 20 = 0$, the terms in y' and the constant term vanish. These equations give $k = -2$, $h = 3$. The new equation then is

$$y'^2 = -8x',$$

which is of the form $y^2 = -2px$. The curve is a parabola which has its vertex at $(3, -2)$ and its axis $y = -2$.



PROBLEMS

1. Find the center of symmetry by solving for x and y, and move the origin to this point. Draw the curve and both sets of axes.

(a) $9x^2 - 36x + 4y^2 + 16y + 16 = 0$;
(b) $16x^2 - 64x - 25y^2 - 336 = 0$;
(c) $4x^2 - y^2 + 32x + 4y + 60 = 0$;
(d) $x^2 + y^2 - 18x + 6y + 54 = 0$;
(e) $x^2 + x + 3y^2 - 9y + 4 = 0$;
(f) $6x^2 - 4y^2 - 12x + 16y + 34 = 0$.

2. Simplify the following equations by the second method. Draw the curve and both sets of axes.

(a) $4x^2 + y^2 + 16x - 6y = 12$;
(b) $16x^2 - y^2 + 32x - 6y = 0$;
(c) $4x^2 - 4x - 3y - 3 = 0$;
(d) $y^2 + 6y - 3x + 6 = 0$;
(e) $3x^2 - 18x + 4y^2 - 8y - 5 = 0$;
(f) $4x^2 + 16x + 3y^2 - 12y + 12 = 0$;
(g) $y = x^3 + 9x^2 + 27x + 27$;
(h) $y = x^3 - 4x^2 + 3x$.

3. Find the equation of each of the following conics and translate the origin to the center of symmetry:

(*a*) Focus (4, 0), directrix $x = 0$, $e = \frac{2}{3}$;
(*b*) focus (4, 0), directrix $x = 0$, $e = \frac{3}{2}$;
(*c*) focus (1, 2), directrix $x = -4$, $e = \frac{1}{2}$;
(*d*) focus (2, 2), directrix $x = -4$, $e = 2$;
(*e*) focus (3, − 4), directrix $y = 0$, $e = \frac{2}{3}$;
(*f*) focus (3, − 4), directrix $y = 0$, $e = \frac{3}{2}$;
(*g*) focus (− 2, 3), directrix $y = 1$, $e = \frac{1}{2}$;
(*h*) focus (− 2, 3), directrix $y = 1$, $e = 2$.

Ans. (*a*) $25x^2 + 45y^2 = 576$.

4. Find the equation of each of the following parabolas and translate the origin to the vertex:

(*a*) Focus (4, 0), directrix $x = 0$;
(*b*) focus (1, 2), directrix $x = -4$;
(*c*) focus (3, − 4), directrix $y = 0$;
(*d*) focus (− 2, 3), directrix $y = 1$;
(*e*) focus (0, 0), vertex (0, 2);
(*f*) focus (− 2, 3), vertex (3, 3);
(*g*) focus (2, 1), vertex (1, 1).

Ans. (*a*) $y^2 = 8x$.

5. Show that the equation of the conic, with the origin at the focus and the principal axis as the x-axis, is $x^2 + y^2 = e^2(x + p)^2$.

Transform this to the standard forms as in § 87 for

(*a*) the parabola; (*b*) the ellipse; (*c*) the hyperbola.

In each of the following locus problems simplify the equation by a translation of the axes:

6. Find the locus of the middle points of chords of the ellipse $36x^2 + 9y^2 = 144$ drawn from one end of:

(*a*) the major axis; (*b*) the minor axis.

7. Find the locus of the centers of all circles tangent to the circle $x^2 + y^2 = a^2$ and the line $y = b$.

8. From one focus of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, focal radii are drawn and bisected. Show that the locus of the points of bisection is an ellipse, and find its center and foci.

90. Discussion of the Equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$. This represents the general form of a quadratic in x and y with the xy term lacking. The only restriction on

the coefficients is that A and C cannot both be zero. We desire to find under what conditions the locus is a parabola, an ellipse, or a hyperbola.

Case 1. *When either A or C is zero.* Suppose $A = 0$. Here the equation has the form

$$Cy^2 + Dx + Ey + F = 0.$$

If $D \neq 0$, by a proper translation we can remove* the y term and the constant, and obtain an equation of the form $Cy'^2 + Dx' = 0$, of which the locus is a parabola.

If $D = 0$, the y term can be removed by translation, and we get $Cy'^2 + F' = 0$, of which the locus is imaginary if F' and C have the same sign; the new x-axis if $F' = 0$; and a pair of lines parallel to the x-axis if F' and C have unlike signs.

A similar discussion holds when $C = 0$.

Case 2. *When A and C are of like sign.* Removing the first degree terms by translation, we have

$$Ax'^2 + Cy'^2 = F'.$$

This is evidently an ellipse if F' has the same sign as A and C; a point if $F' = 0$; and an imaginary locus if F' has the opposite sign to that of A and C.

Case 3. *When A and C are of unlike sign.* Removing the first-degree terms, we have

$$Ax'^2 + Cy'^2 = F'.$$

This is evidently a hyperbola unless $F' = 0$, in which case the locus is a pair of lines intersecting at the new origin.

* By actual substitution of $x = x' + h$ and $y = y' + k$, it can be shown that (a) translation does not affect the coefficients of the highest powers of the variables; (b) that, if neither A nor C is zero, the first-degree terms can be removed; (c) that, if one variable appears only to the first degree, the first power of the other variable and the constant term can be removed; (d) that, if one variable is missing, the first power of the other can be removed.

91. Limiting Forms. — We have seen in §§ 58, 70, 83 how the various conics may be formed by passing a plane through a cone. In the previous section we have found that in general the locus of the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is a conic, but that for particular values of the coefficients the locus is imaginary, or is a point, a line, or a pair of lines. These degenerate cases may be regarded as limiting forms of sections of a cone. The discussion of the different cases is as follows:

(*a*) If the vertex of the cone is moved away from the base indefinitely, the cone approaches the cylinder as a limiting form. Here the elements are parallel and the parabolic section approaches a pair of straight lines. If the cutting plane becomes a tangent, the limit is a double straight line. If the plane is moved farther, there is no intersection, or an imaginary one.

(*b*) If the plane forming a circular or elliptical section is moved towards the vertex, the section grows smaller, until at the vertex it becomes a point. If it is moved still farther, there is no intersection, or an imaginary one.

(*c*) Any section of a cone formed by a plane passing through an element is a triangle, hence, if the plane forming a hyperbolic section is moved to the vertex, the section becomes two intersecting straight lines.

92. General Statement. — The preceding analysis may be summed up as follows:

A quadratic equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

always defines a conic or one of its limiting forms, the coefficients being any real numbers, zero included.

If either A or C is zero, the conic is a parabola or in special cases two parallel straight lines distinct, coincident, or imaginary.

If A and C have the same sign, the conic is an ellipse, or in special cases a circle, point, or imaginary ellipse.

If A and C have unlike signs, the conic is a hyperbola, or in special cases two intersecting straight lines.

93. Generalized Standard Equations of the Conics. — The equations derived in Chapters IV, V, and VI we have seen to be of a special character, since they are applicable only to the case that the vertex or center is at the origin and the principal axis is the x- or y-axis. We are now in a position to generalize these equations and obtain standard forms where the only restriction is that the principal axis is parallel to one of the coördinate axes.

Consider an ellipse with the center at (h, k), semi-axes a and b, and transverse axis $y = k$. It is required to find its equation.

If the origin is translated to (h, k), we know from Chapter V that the transformed equation will be

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1.$$

The problem then is the reverse of that considered in § 89, viz., the transformed equation is given and the equation with reference to the old axes is to be found. Hence the substitutions are those of § 85 reversed, that is, $x' = x - h$, and $y' = y - k$. These give

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

as the required equation.

In the same way we discuss the other cases and obtain the following equations:

Parabola —

Vertex (h, k), axis $y = k$: $(y-k)^2 = 2p(x-h)$; (30)

" " " $x = h$: $(x-h)^2 = 2p(y-k)$; (30a)

Ellipse —

Center (h, k), axis $y = k$: $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1; \qquad (31)$$

" " " $x = h$: $$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1; \qquad (31a)$$

Hyperbola —

Center (h, k), axis $y = k$: $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1; \qquad (32)$$

" " " $x = h$: $$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \qquad (32a)$$

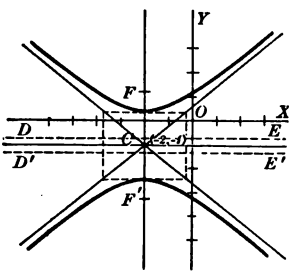
These forms are easy to remember if one bears in mind that they are merely generalizations of the simple forms $y^2 = 2px$, $x^2 = 2py$, etc. To reduce an equation to one of these forms, it is necessary only to complete the squares.

Example 1. — Write the equation of the ellipse which has the center $(0, 1)$, one focus $(-4, 1)$, and minor axis 6.

Solution. — Here the principal axis is $y = 1$. From the given data $b = 3$ and $c = 4$, whence $a = 5$. Therefore the equation is

$$\frac{(x-0)^2}{25} + \frac{(y-1)^2}{9} = 1.$$

Example 2. — Simplify the equation $2x^2 - 3y^2 + 8x - 6y + 11 = 0$. Find the center, semi-axes, vertices, and foci, and draw the curve.



Solution. — Completing squares,

$$2(x+2)^2 - 3(y+1)^2 = -6$$

or $$-\frac{(x+2)^2}{3} + \frac{(y+1)^2}{2} = 1.$$

The center is $(-2, -1)$ and since the term in $(y+1)^2$ is positive, the transverse axis is $x = -2$. Hence $a = \sqrt{2}$, $b = \sqrt{3}$, and $c = \sqrt{5}$. The vertices are

$$(-2, -1 \pm \sqrt{2})$$

and the foci $(-2, -1 \pm \sqrt{5})$.

PROBLEMS

1. Write the equations of the following ellipses in the general form, the major axis being first parallel to the x-axis, and second parallel to the y-axis, unless the given data fix its direction,

(*a*) $a = 5$, $b = 3$, center $(2, 3)$;
(*b*) $a = 4$, $b = 2$, center $(-1, 2)$;
(*c*) $a = \frac{2}{5}$, $b = \frac{1}{5}$, center $(-2, 3)$;
(*d*) vertices $(\pm 5, 3)$, focus $(3, 3)$;
(*e*) vertices $(-3, 1)$, $(7, 1)$, focus $(0, 1)$;
(*f*) foci $(\pm 3, -2)$, directrix $x = -4$;
(*g*) minor axis 12, $e = \frac{5}{13}$, focus $(0, 0)$;
(*h*) major axis 6, $e = \frac{1}{5}$, focus $(1, 1)$;

Ans. (*d*) $\dfrac{x^2}{25} + \dfrac{(y-3)^2}{16} = 1$;

(*g*) $\dfrac{(x \pm \frac{5}{2})^2}{\frac{169}{4}} + \dfrac{y^2}{36} = 1.$

2. Write the equations of the hyperbolas satisfying the data of Problem 1, if possible. If not possible, explain why and revise data to make a similar problem for the hyperbola and solve.

Ans. (*d*) not possible, for $c < a$. Let the vertices be $(\pm 3, 3)$ and the focus $(5, 3)$. Then the solution is $\dfrac{x^2}{9} - \dfrac{(y-3)^2}{16} = 1.$

3. Write by inspection the equations of the parabolas having:

(*a*) focus $(0, 0)$, vertex $(2, 0)$; (*c*) focus $(-2, 3)$, vertex $(3, 3)$.
(*b*) focus $(2, 1)$, vertex $(2, 5)$;

4. Simplify the following equations by completing the squares and draw their loci:

(*a*) $3x^2 - 2y^2 - 6x - 8y - 11 = 0$; (*f*) $3y = (x-1)(x+2)$;
(*b*) $x^2 + 4x - 5y - 6 = 0$; (*g*) $y^2 = (x-1)(x+2)$;
(*c*) $4x^2 - 9y^2 + 12x + 6y = 28$; (*h*) $x^2 + 4y^2 + 6x + 4y = 0$;
(*d*) $9y^2 - 12y - 2x - 2 = 0$; (*i*) $4x^2 + y^2 + 4x + 6y = 0$;
(*e*) $9x^2 + 4y^2 - 36x + 4y + 1 = 0$; (*j*) $(2x-1)^2 + (3y-2)^2 = 9y^2$.

5. Write the solutions of Problems 6, 7, and 8, page 114, in standard form.

6. The base of a triangle is fixed in length and position. Find the locus of the vertex if one base angle is twice the other.

94. Simplification of Equations by Rotation. — The process of rotation of the axes effects the removal of the xy term

from the second-degree equation if the proper angle θ is chosen. Let us substitute in the general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

the rotation formulas

$$x = x' \cos\theta - y' \sin\theta,$$
$$y = x' \sin\theta + y' \cos\theta.$$

The coefficient of the $x'y'$ term will be

$$-2A \sin\theta \cos\theta + B(\cos^2\theta - \sin^2\theta) + 2C \sin\theta\cos\theta$$

or $$-A \sin 2\theta + B\cos 2\theta + C \sin 2\theta.$$

(The student should make the substitution in full for all terms of the general equation.)

The term in $x'y'$ will vanish if

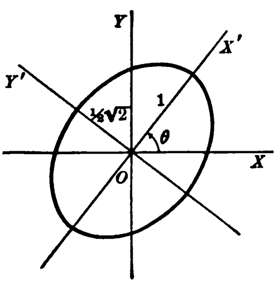
$$(C - A)\sin 2\theta + B\cos 2\theta = 0, \quad \text{or} \quad \tan 2\theta = \frac{B}{A - C}. \qquad (33)$$

Thus we have the theorem:

For any second degree equation there is an angle of value less than 90° such that the substitutions for rotation through this angle will transform the equation into one containing no xy term.

For the equation $\tan 2\theta = \dfrac{B}{A - C}$ gives values of $\tan 2\theta$, positive or negative, from 0 when $B = 0$ to infinity when $A = C$. Hence for all values of A, B, and C, 2θ will have some value between 0 and 180°.

EXAMPLE. — Simplify $41x^2 - 24xy + 34y^2 = 25$.



Here $A = 41$, $B = -24$, and $C = 34$.

Therefore $\tan 2\theta = \dfrac{-24}{7}$.

This gives at once

$$\cos 2\theta = \frac{-7}{25}.$$

Substituting in the half angle formulas of trigonometry,

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \frac{4}{5},$$

and $$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \frac{3}{5}.$$

Hence $$x = \frac{3}{5}x' - \frac{4}{5}y' \text{ and } y = \frac{4}{5}x' + \frac{3}{5}y'.$$

Substituting in the above equation and reducing, we get

$$x'^2 + 2y'^2 = 1,$$

an ellipse of semi-axes 1 and $\frac{\sqrt{2}}{2}$.

95. The General Equation of the Second Degree. — We have seen that the equations of the conics are all special cases of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

We are now ready to prove the converse theorem; namely,

Every equation of the second degree in two variables defines a conic or one of the limiting forms of the conic.

It has been shown in § 90 that this theorem is true for every form of the second-degree equation in which the product xy does not appear. Also we have just seen in the previous section that by a proper rotation of the axes the xy term can be made to disappear from the equation. Thus the proof of the theorem is complete.

96. The Characteristic. — The quantity $B^2 - 4AC$ is called the characteristic of the general second-degree equation, and is denoted by Δ. We now prove the theorem:

The characteristic of a general equation of the second degree is unaltered by a rotation of the axes through any angle θ.

In the general equation substitute the values given in the rotation formulas. The first three terms become

$$\begin{array}{r|l r|l r|l} A\cos^2\theta & x'^2 & -2A\sin\theta\cos\theta & x'y' & +A\sin^2\theta & y'^2 \\ +B\sin\theta\cos\theta & & -B\sin^2\theta & & -B\sin\theta\cos\theta & \\ & & +B\cos^2\theta & & & \\ +C\sin^2\theta & & +2C\sin\theta\cos\theta & & +C\cos^2\theta & \end{array}$$

Calling the coefficients of the new equation A', B', C', etc., we have:

$$A' = A\cos^2\theta + C\sin^2\theta + B\sin\theta\cos\theta,$$
$$C' = A\sin^2\theta + C\cos^2\theta - B\sin\theta\cos\theta,$$
$$B' = 2\,C\sin\theta\cos\theta - 2\,A\sin\theta\cos\theta + B\cos^2\theta - B\sin^2\theta.$$

The characteristic of the new equation, $B'^2 - 4\,A'C'$, becomes on multiplying and collecting the terms:

$$B^2\cos^4\theta + B^2\sin^4\theta - 8\,AC\sin^2\theta\cos^2\theta - 4\,AC\sin^4\theta - 4\,AC\cos^4\theta + 2\,B^2\sin^2\theta\cos^2\theta$$
$$= (B^2 - 4\,AC)\cos^4\theta + (B^2 - 4\,AC)\sin^4\theta + 2(B^2 - 4\,AC)\sin^2\theta\cos^2\theta$$
$$= (B^2 - 4\,AC)(\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta)$$
$$= (B^2 - 4\,AC)(\cos^2\theta + \sin^2\theta)^2$$
$$= B^2 - 4\,AC.$$

It is easy to see that Δ is also unchanged by a translation of the axes. For this reason it is called an *invariant* of the equation.

Exercise. — Prove that $A + C$ is not changed by the substitutions for rotation or translation of the axes.

97. Test for distinguishing the Conics. — If in the previous section, the angle θ were chosen so that the xy term vanished, the new equation would be

$$A'x'^2 + C'y'^2 + Dx' + Ey' + F = 0$$

and $$\Delta = B^2 - 4\,AC = B'^2 - 4\,A'C' = -4A'C',$$

since $B' = 0$.

From this relation and § 92 we see that

if $\Delta < 0$, *A' and C' are of like sign, and the conic is an ellipse,**

if $\Delta = 0$, *A' or C' is zero, and the conic is a parabola,**

if $\Delta > 0$, *A' and C' are of unlike sign, and the conic is a hyperbola.**

* Or one of the limiting forms discussed in §§ 90, 91.

98. Suggestions for simplifying the Equation of a Conic. — If the conic is an ellipse or hyperbola (that is, if $\Delta \neq 0$), determine the coördinates (h, k) of the center of symmetry and remove the first-degree terms by translation. Then rotate the axes as in the example of § 94.

If the conic is a parabola ($\Delta = 0$), the substitutions for rotation should be made first, and then equations of condition involving h and k can be formed.

A convenient check on the accuracy of the rotation substitutions is the fact that $A + C$ is unchanged until the new equation is simplified. (See Exercise, § 96.)

PROBLEMS

Simplify the following equations. In each case draw the conic and the three sets of coördinates axes.

1. $73x^2 + 72xy + 52y^2 - 218x - 176y + 97 = 0.$
   Ans. $4x''^2 + y''^2 = 4.$
2. $4x^2 - 24xy + 11y^2 + 72x - 116y + 204 = 0.$
   Ans. $x''^2 - 4y''^2 = -4.$
3. $18x^2 - 48xy + 32y^2 - 120x + 35y + 200 = 0.$
   Ans. $2y''^2 = 3x''.$
4. $6x^2 + 13xy + 6y^2 - 7x - 8y + 2 = 0.$ *Ans.* Two lines.
5. $2x^2 + 4xy + 4y^2 + 2x + 3 = 0.$ *Ans.* Imaginary.
6. $41x^2 + 24xy + 34y^2 - 90x + 120y + 225 = 0.$ *Ans.* Point.
7. $16x^2 + 24xy + 9y^2 - 60x - 45y + 50 = 0.$ *Ans.* $4x''^2 = 1.$
8. $66x^2 + 24xy + 59y^2 + 396x + 72y + 444 = 0.$
   Ans. $3x''^2 + 2y''^2 = 6.$
9. $x^2 + 24xy - 6y^2 + 4x + 48y + 34 = 0.$
   Ans. $3y''^2 - 2x''^2 = 6.$
10. $16x^2 + 24xy + 9y^2 - 60x + 80y + 200 = 0.$
    Ans. $x''^2 + 4y'' = 0.$
11. $7x^2 - 48xy - 7y^2 = 0.$ *Ans.* $x''^2 - y''^2 = 0.$
12. $2x^2 + 6xy + 10y^2 - 2x - 6y + 19 = 0.$ *Ans.* Imaginary.
13. $9x^2 - 24xy + 16y^2 - 3x + 4y - 6 = 0.$ *Ans.* Two lines.

14. $13x^2 + 10xy + 13y^2 - 42x + 6y - 27 = 0.$
Ans. $9x''^2 + 4y''^2 = 36.$

15. $13x^2 + 10xy + 13y^2 + 16x - 16y + 16 = 0.$ *Ans.* Point.

16. $x^2 + 4xy + y^2 - 2x - 10y - 11 = 0.$ *Ans.* $3x''^2 - y''^2 = 9.$

17. $3x^2 + 8xy - 3y^2 - 10x - 30y + 20 = 0.$
Ans. $y''^2 - x''^2 = 4.$

18. $6x^2 - 4xy + 9y^2 - 40x + 30y + 55 = 0.$
Ans. $x''^2 + 2y''^2 = 2.$

19. $3x^2 - 2\sqrt{3}xy + y^2 - (8 + 12\sqrt{3})x + (12 - 8\sqrt{3})y - 12 = 0.$
Ans. $y''^2 = 4x.$

20. The ends of the base of a triangle are (0, 0) and (4, 0). Find the locus of the vertex if the sum of the slopes of the sides is (*a*) $\frac{7}{12}$, (*b*) 1, (*c*) $-\frac{2}{\sqrt{3}}$. Simplify the equation found and draw the locus.

21. Solve the previous problem when the given points are (0, 0) and (*a*, 0) and the sum of the slopes is any constant *k*. Then show that the locus is a hyperbola passing through the given points and having its axes of symmetry inclined at an angle $\frac{1}{2}$ arc tan $\frac{-2}{k}$.

22. The difference of the base angles of the triangle whose base joins (0, 0) and (4, 0) is 45°. Find the locus of the vertex.
Ans. $x''^2 - y''^2 = 2\sqrt{2}$ or $x''^2 - y''^2 = -2\sqrt{2}.$

99. The Conic through Five Points. — The general equation of the second degree involves six arbitrary constants. As in the case of the circle, however, one of these can be divided out; hence, if we can express five of the coefficients in terms of the sixth, the conic is completely determined. Geometrically, this means that the conic is completely determined by five points, or by five other geometric conditions.

To find the equation of a conic so defined we proceed to form equations between the coefficients of the general equation and solve them. For example, suppose the conic is to pass through the points (4, 2), (2, 4), (– 3, 1), (1, – 3), and (0, 0). Substituting these in the general equation,

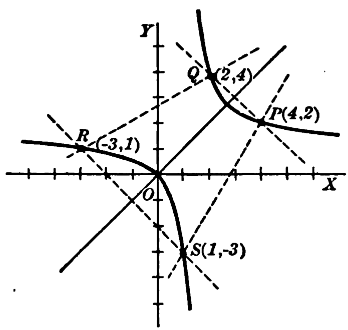
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

we have:
$$16A + 8B + 4C + 4D + 2E + F = 0,$$
$$4A + 8B + 16C + 2D + 4E + F = 0,$$
$$9A - 3B + C - 3D + E + F = 0,$$
$$A - 3B + 9C + D - 3E + F = 0,$$
$$F = 0.$$

Solving these, we find $A = C = \dfrac{B}{50}$, $D = E = -\dfrac{7B}{5}$. This makes the general equation, on dividing out the B and clearing of fractions:

$$x^2 + 50xy + y^2 - 70x - 70y = 0.$$

The value of Δ shows that the conic is a hyperbola. Its center is found by the method of § 89 to be $(\frac{35}{26}, \frac{35}{26})$. The inclination of its axes is 45°, since $\tan 2\theta = \infty$. This information, together with the five points given, is sufficient for drawing the curve. The graph follows.



A shorter method of solving the above problem is given by the theorem of § 41. Call the first four points P, Q, R, and S. The equations of the lines PQ and RS are $x + y - 6 = 0$ and $x + y + 2 = 0$. Multiplying these together, the equation of the pair of lines is

$$x^2 + 2xy + y^2 - 4x - 4y - 12 = 0,$$

by § 42. Similarly the equation of the pair of lines RQ and SP is

$$15x^2 - 34xy + 15y^2 + 28x + 28y - 196 = 0.$$

Now the intersections of these two equations are the points P, Q, R, and S. Hence by § 41

$$15\,x^2 - 34\,xy + 15\,y^2 + 28\,x + 28\,y - 196$$
$$+ k(x^2 + 2\,xy + y^2 - 4\,x - 4\,y - 12) = 0 \quad (a)$$

is the equation of a system of curves passing through these four points. As this is of the second degree for all values of k, they are all conics. To find the conic of this system passing through the fifth point (0, 0), substitute its coördinates for the variables. This gives

$$-196 - 12\,k = 0,$$

whence $k = -\frac{49}{3}$. Putting this value for k in (a) and collecting terms, we have the same result as before.

PROBLEMS

1. Find the equation of the conic passing through the following points:

(a) (3, 2), (1, 2), (3, 0), $(2, \frac{9}{4})$, $(\frac{11}{3}, 1)$;
(b) (1, 6), (− 3, − 2), (− 5, 0), (3, 4), (0, 10);
(c) (3, 1), (3, 5), (6, 2), (6, 10), (11, 5);
(d) (1, 1), (− 1, 2), (0, − 2), (− 2, − 1), (3, −3).

2. Find the equation of the parabola passing through the following points:

(a) (0, 0), (4, 0), (0, 3), (− 8, 8);
(b) (0, 1), (4, 7), (4, 1), (12, 7);
(c) (1, 1), (4, 0), (0, 4), (9, 1);

Hint. — One of the conditions is $B^2 - 4AC = 0$.

3. Find the equation of a conic with axes of symmetry parallel to the coördinate axes and passing through:

(a) (− 4, 5), (6, 0), (−4, −5), (4, 3);
(b) (7, 1), (6, − 2), (7, −5), (− 2, − 2);
(c) (− 3, − 4), (− 1, − 3), (1, 0), (3, 5).

4. Find the equation of a conic symmetrical with respect to the origin and passing through:

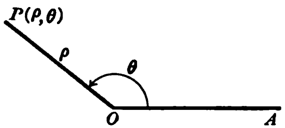
(a) (2, 4), (3, 1), (− 2, 6);
(b) (1, 2), (− 1, 1), (3, 2);
(c) (0, 0), (6, 4), (4, 6).

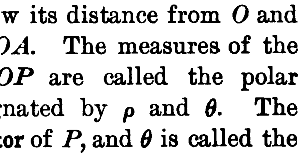
CHAPTER VIII

POLAR COÖRDINATES

100. Definition. — Some topics in analytic geometry can be better investigated by the use of *polar coördinates* than by rectangular coördinates. In the polar system the position of a point is fixed by measuring a distance and a direction instead of by the measures of two distances. This is essentially the same system as that of bearing and distance used in surveying, contrasted with that of latitude and longitude used in geography.

Choose a fixed point O as the origin, called the **pole**, and a fixed line OA through it, called the **polar axis**. Then any





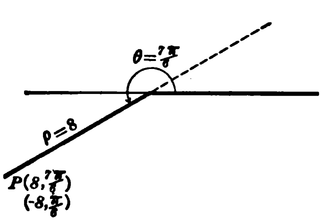
point P is determined if we know its distance from O and the angle that OP makes with OA. The measures of the distance OP and the angle AOP are called the polar coördinates of P and are designated by ρ and θ. The distance ρ is called the **radius vector** of P, and θ is called the **vectorial angle.**

Polar coördinates do not obey the conventions of the rectangular system as to their direction and magnitude. As in trigonometry the radius vector may be rotated indefinitely in a counter-clockwise or clockwise direction, making θ take on any positive or negative value. *Distances measured on the terminal line of θ away from the pole are*

positive; those measured in the opposite direction, on the terminal produced, are negative. (See the right-hand diagram on page 127.) Hence every pair of real numbers (ρ, θ) determines one point which may be located according to the following rule.

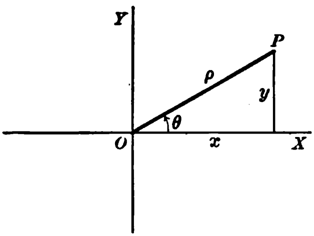
RULE FOR PLOTTING. — *Taking the polar axis as an initial line, lay off the vectorial angle θ, counter-clockwise if positive, clockwise if negative. Then measure off the radius vector ρ, on the terminal of θ if positive, on the terminal of θ produced through the pole if negative.*

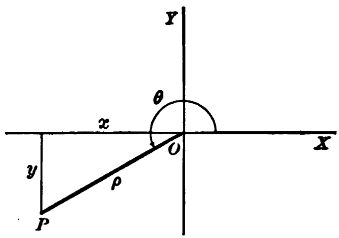
Since θ and $\theta + 2\pi$ have the same terminal line, a point may be represented by an indefinite number of pairs of coördinates. Thus in the adjoining figure we may take for the coördinates of P, $\rho = 8$, $\theta = \frac{7\pi}{6}$ or $\frac{7\pi}{6} \pm 2n\pi$. Since the terminal line of θ produced is the terminal of $\theta \pm \pi$, a second set of coördinates is



$\rho = -8$, $\theta = \frac{\pi}{6}$ or $\frac{\pi}{6} \pm 2n\pi$. Ordinarily we keep θ within the limits $\pm\pi$, or 0 and 2π.

101. Relations between Rectangular and Polar Coördinates. — Take the pole at the origin of rectangular coördinates and the polar axis as the positive half of the x-axis. From





the figures and the definitions of the trigonometric functions it is evident that for $P(x, y) = P(\rho, \theta)$ in any quadrant the following formulas are true:

$$x = \rho \cos \theta, \qquad y = \rho \sin \theta, \tag{34}$$

$$\rho^2 = x^2 + y^2, \qquad \theta = \text{arc} \tan \frac{y}{x}. \tag{35}$$

These equations enable us to transform rectangular equations and coördinates into polar forms, and conversely.

NOTE. — In transforming the rectangular coördinates of a point into polar coördinates, care should be taken to group together the corresponding values of ρ and θ.

102. Polar Curves. — The *definitions* of equation and locus in polar are the same as those in rectangular coördinates, if (ρ, θ) is substituted for (x, y) (§ 14). The equation in polar coördinates is derived as in the rectangular system (§ 16). In a few cases the polar equation may be obtained best by deriving it in the rectangular form and then substituting for x and y their values in terms of ρ and θ and *vice versa.*

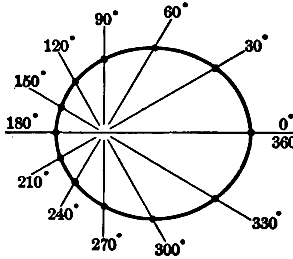
Plotting in polar coördinates resembles that in rectangular coördinates. The equation should usually be solved for ρ and a table of values formed, taking values of θ at intervals of 30° (or sometimes 15°). When the curve has symmetry, it is usually unnecessary to carry the table through more than two quadrants.

EXAMPLE. —Plot the ellipse $2\rho - \rho \cos \theta = 6$.

Solving, $\rho = \dfrac{6}{2 - \cos \theta}$.

TABLE OF VALUES

$\theta°$	ρ	$\theta°$	ρ
0	6.0	210	2.1
30	5.3	240	2.4
60	4.0	270	3.0
90	3.0	300	4.0
120	2.4	330	5.3
150	2.1	360	6.0
180	2.0		



PROBLEMS

1. Plot the points: (*a*) $(6, \pm 30°)$; (*b*) $(\pm 10, 420°)$; (*c*) $\left(8, \frac{\pi}{6}\right)$ $\left(8, \frac{5\pi}{6}\right)$. What symmetry has each pair of points?

2. Plot $\left(8, \frac{2\pi}{3}\right)$. Plot points symmetrical to this with respect to the pole, polar axis, and the 90° axis, and find their coördinates.

3. As in Problem 2 find the points symmetrical to:
(*a*) $\left(16, \frac{4\pi}{3}\right)$; (*b*) $\left(-6, -\frac{2\pi}{3}\right)$; (*c*) $\left(-10, -\frac{3\pi}{2}\right)$.

4. Fix $P(\rho, \theta)$ in any quadrant and find its symmetry to:
(*a*) $P(\rho, -\theta)$; (*b*) $P(-\rho, \theta)$; (*c*) $P(\rho, \pi - \theta)$.

5. Find the rectangular coördinates of the following points:
(*a*) $\left(16, \frac{\pi}{6}\right)$; (*b*) $\left(4, \frac{2\pi}{3}\right)$; (*c*) $\left(6, \frac{\pi}{2}\right)$;
(*d*) $(4, \pi)$; (*e*) $\left(-2, \frac{\pi}{4}\right)$; (*f*) $\left(-8, \frac{3\pi}{4}\right)$.

Ans. (*a*) $(8\sqrt{3}, 8)$.

6. Find two pairs of polar coördinates for each of the following points and plot the point in each case:
(*a*) $(-1, \sqrt{3})$; (*b*) $(2, -2)$; (*c*) $(5, -12)$;
(*d*) $(2, -2\sqrt{3})$; (*e*) $(-\sqrt{2}, -\sqrt{2})$.

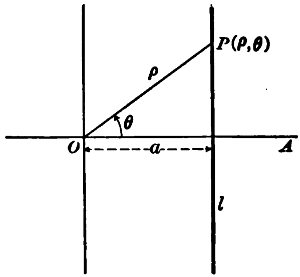
Ans. (*a*) $(2, 120°)$, $(-2, 300°)$.

7. Plot each of the following equations and identify each curve by transforming its equation to rectangular coördinates:
(*a*) $\rho = 4$; (*d*) $\rho \cos \theta = -2$; (*g*) $\rho = 2a \cos \theta$;
(*b*) $\theta = \text{arc} \tan \frac{3}{4}$; (*e*) $\rho = \pm 6$; (*h*) $\rho \sin \theta = 2a$.
(*c*) $\rho = 2 \cos \theta$; (*f*) $\tan \theta = \frac{2}{3}$.

8. Draw each of the following curves and transform its equation into polar coördinates:
(*a*) $y = 3x$; (*g*) $x^2 + y^2 + 4y = 0$;
(*b*) $3x + 2y = 0$; (*h*) $x^2 - y^2 = 16$;
(*c*) $x - a = 0$; (*i*) $y^2 - x^2 = 16$;
(*d*) $y = -3$; (*j*) $x^2 - y^2 = a^2$;
(*e*) $x^2 + y^2 = a^2$; (*k*) $2xy = a^2$;
(*f*) $x^2 + y^2 - 4x = 0$; (*l*) $2xy = -a^2$.

Ans. (*a*) $\tan \theta = 3$; (*c*) $\rho \cos \theta = a$; (*f*) $\rho = 4 \cos \theta$; (*h*) $\rho^2 \cos 2\theta = 16$; (*k*) $\rho^2 \sin 2\theta = a^2$.

103. The Equation of the Straight Line. — The general equation of the straight line in polar coördinates is not as convenient as the equation in rectangular coördinates and will not be discussed. The special cases where the line is parallel or perpendicular to the polar axis or passes through the pole lead to very simple equations.



In the figure let the line l be perpendicular to the polar axis OA and have the polar intercept a. For any point $P(\rho, \theta)$ on the line it is evident that $\cos\theta = \frac{a}{\rho}$, whence the equation of the line is

$$\rho \cos \theta = a. \tag{36}$$

Similarly the equation of a line parallel to the polar axis, of 90° intercept a, is

$$\rho \sin \theta = a. \tag{36a}$$

For a line passing through the pole the equation is evidently

$$\theta = c. \tag{37}$$

Exercise 1. By transforming the normal form of the straight-line equation into polar coördinates show that the equation of any line in polar coördinates is

$$\rho \cos(\theta - \omega) = p.$$

104. The Equation of the Circle. — As in the case of the straight line the general form of the circle equation is not often used. Several special forms, which are common, are:

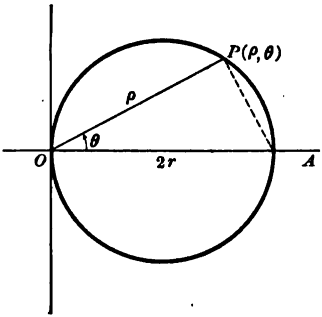
Circle with center at pole, radius r: $\rho = r$; (38)

Circle with center at $(r, 0)$, radius r: $\rho = 2r\cos\theta$; (39)

Circle with center at $\left(r, \frac{\pi}{2}\right)$, radius r: $\rho = 2r\sin\theta$; (39a)

Circle through pole, with polar intercept a and 90° intercept b:

$$\rho = a \cos \theta + b \sin \theta. \qquad (40)$$



The first equation is obvious. The second may be readily derived from the adjoining figure, and the third from a similar figure.

The simplest way of getting the fourth equation is to observe that its rectangular equation is $x^2 + y^2 - ax - by = 0$. Transforming this into polar coördinates, we have the desired result.

Exercise 2. Derive equation (40) directly from a figure.

Hint. — Show that the radius vector is in length the sum of the projections upon it of the x- and y- intercepts of the circle.

105. Discussion of Polar Curves. — As in the case of rectangular equations, the study of polar curves is facilitated by discussion of the equation in conjunction with plotting. The topics discussed are of much the same character.

(*a*) *Intercepts.* — The intercepts on the polar and 90° axes are the values of ρ for $\theta = 0°$, 90°, 180°, and 270°.

(*b*) *Symmetry.* — It is easy to establish that in accordance with the definitions of § 22, (ρ, θ) is symmetrical to $(-\rho, \theta)$ with respect to the pole; to $(\rho, -\theta)$ with respect to the polar axis; and to $(\rho, \pi - \theta)$ with respect to the 90° axis. Hence we have the following tests for symmetry:

If the substitution of

$$\left.\begin{array}{r} -\rho \textit{ for } \rho \\ -\theta \textit{ for } \theta \\ \pi - \theta \textit{ for } \theta \end{array}\right\} \textit{does not change the form of the equation, there is symmetry with respect to} \left\{\begin{array}{l} \textit{the pole;} \\ \textit{the polar axis;} \\ \textit{the 90° axis.} \end{array}\right.$$

Exercise 3. Prove that (ρ, θ) is symmetrical to:

(*a*) $(-\rho, \theta)$ with respect to the pole;

(*b*) $(\rho, -\theta)$ with respect to the polar axis;

(*c*) $(\rho, \pi - \theta)$ with respect to the 90° axis;

(*d*) $(\rho, \pi + \theta)$ with respect to the pole*;

(*e*) $(-\rho, \pi - \theta)$ with respect to the polar axis*;

(*f*) $(-\rho, -\theta)$ with respect to the 90° axis.*

(*c*) *Limiting Values of* θ. — If the solution for ρ gives $\rho = a \pm \sqrt{f(\theta)}$, the equation $f(\theta) = 0$ usually determines limiting values of θ, since values of θ making $f(\theta) < 0$ make ρ imaginary and must be excluded.

(*d*) *Extent of the Curve.* — If any values of θ make ρ infinite, these values determine the direction in which the curve extends to infinity. If ρ is never infinite, the values of θ which make ρ take on its greatest numerical value should be found. Likewise the values of θ for which $\rho = 0$, or for which ρ takes on its least numerical value, should be found.

Example 1. — Discuss and plot the locus of the equation $\rho = \dfrac{2}{1 + \cos\theta}$.

(*a*) Intercepts:

θ	0°	90°	180°	270°
ρ	1	2	∞	2

(*b*) Symmetry: With respect to the polar axis, since $\cos(-\theta) = \cos\theta$. Hence the table of values is unnecessary for $\theta > 180°$.

(*c*) Limiting values of θ: None, since no radical is involved.

(*d*) Extent: ρ is infinite when $1 + \cos\theta = 0$, *i.e.* when $\cos\theta = -1$, or $\theta = \pi$.

ρ is evidently never 0. Its least value occurs when $1 + \cos\theta$ is greatest, *i.e.* when $\cos\theta = 1$, or $\theta = 0$. Here $\rho = 1$.

* Exercise 3, *d*, *e*, and *f*, gives three alternative tests for symmetry. Their existence is due to the double representation of a point in polar coördinates.

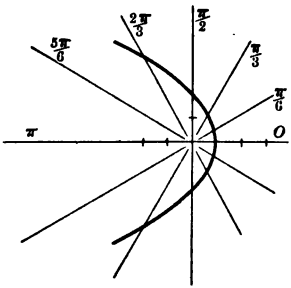


TABLE OF VALUES

θ (radians)	θ (degrees)	$1+\cos\theta$	ρ
± 0	0	2.00	1.0
$\pm\frac{\pi}{6}$	$\pm$ 30	1.87	1.1
$\pm\frac{\pi}{3}$	$\pm$ 60	1.50	1.3
$\pm\frac{\pi}{2}$	$\pm$ 90	1.00	2.0
$\pm\frac{2\pi}{3}$	$\pm$ 120	0.50	4.0
$\pm\frac{5\pi}{6}$	$\pm$ 150	0.13	15.0
$\pm\pi$	$\pm$ 180	0.00	∞

Transformation to rectangular coördinates shows that this curve is a parabola.

EXAMPLE 2. — $\rho^2 = a^2\cos 2\theta$, or $\rho = \pm a\sqrt{\cos 2\theta}$.

(*a*) The intercepts on the polar axis are $\pm a$; on the 90° axis they are imaginary.

(*b*) The three tests for symmetry are satisfied, for $\cos 2(-\theta) = \cos 2\theta$, etc. Hence values of θ up to 90° only are needed.

(*c*) ρ is imaginary when $\cos 2\theta < 0$, *i.e.* when

$$90^\circ < 2\theta < 270^\circ,$$

or

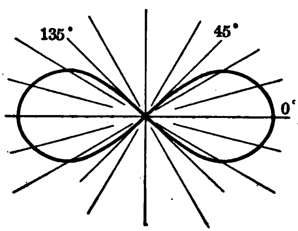
$$45^\circ < \theta < 135^\circ.$$

(*d*) The greatest numerical value of ρ occurs when $\cos 2\theta = 1$, or $\theta = 0$ or 180°. This value is a.

$\rho = 0$ when $\cos 2\theta = 0$, or when $2\theta = 90^\circ$, 270°, etc., or $\theta = 45^\circ$, 135°, 225°, and 315°.

TABLE OF VALUES, $(a = 10)$

θ	2θ	$\cos 2\theta$	$\sqrt{\cos 2\theta}$	ρ
0	0	1.00	$\pm$ 1.00	$\pm$ 10
15°	30	.87	$\pm$ .93	$\pm$ 9.3
30°	60	.50	$\pm$ .71	$\pm$ 7.1
45°	90	.00	.00	0.0



EXAMPLE 3. — $\rho = 1 + 2 \sin \theta$.

(a)

θ	0°	90°	180°	270°
ρ	1	3	1	-1

(b) Since $\sin(\pi - \theta) = \sin \theta$, the curve is symmetrical with respect to the 90° axis.

(c) No values of θ are excluded.

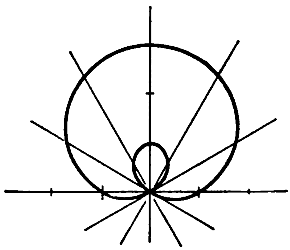
(d) ρ is never infinite; its greatest value occurs when $\sin \theta = 1$, or $\theta = 90°$.

$\rho = 0$ when $1 + 2 \sin \theta = 0$, or $\sin \theta = -\frac{1}{2}$. Hence the curve passes through the pole when $\theta = 210°$ or $330°$.

TABLE OF VALUES

θ	$\sin \theta$	ρ
0	.00	1.00
30	.50	2.00
60	.87	2.73
90	1.00	3.00
120	.87	2.73
150	.50	2.00
180	.00	1.00

θ	$\sin \theta$	ρ
210	$-.50$	.00
240	$-.87$	$-.73$
270	-1.00	-1.00
300	$-.87$	$-.73$
330	$-.50$	.00
360	.00	1.00



PROBLEMS

1. Identify and draw the locus of each of the following:

(a) $\rho \cos \theta - 4 = 0$;
(b) $\rho \cos \theta + 4 = 0$;
(c) $\rho = 4$;
(d) $\tan \theta = \sqrt{3}$;
(e) $\rho - 8 \sin \theta = 0$;
(f) $\rho + 7 \cos \theta = 0$;
(g) $\rho^2 + 3\rho - 4 = 0$;
(h) $\sec^2 \theta = 2$;
(i) $\theta = \frac{\pi}{6}$;
(j) $\rho = -6 \sin \theta$;
(k) $\rho = 3 \cos \theta$;
(l) $\rho^2 - 2\rho(\cos \theta - \sin \theta) - 7 = 0$.

2. Transform into polar coördinates:

(a) $Ax + By + C = 0$;
(b) $x^2 + y^2 + Dx + Ey + F = 0$.

Discuss and plot the locus of each of the following:

3. (a) $\rho = a(1 - \cos \theta)$; (c) $\rho = a(1 - \sin \theta)$;
(b) $\rho = a(1 + \cos \theta)$; (d) $\rho = a(1 + \sin \theta)$. (Cardioids.)

4. (a) $\rho = a + b\cos\theta$; (c) $\rho = a + b\sin\theta$;
(b) $\rho = a - b\cos\theta$; (d) $\rho = a - b\sin\theta$. (Limaçons: three forms according as $a <, =,$ or $> b$.)

5. (a) $\rho = a\tan^2\theta\sec\theta$; (b) $\rho = a\cot^2\theta\csc\theta$. (Semi-cubical parabolas.)

6. (a) $\rho = a\sec\theta \pm b$; (b) $\rho = a\csc\theta \pm b$. (Conchoids of Nicomedes; three forms according as $a <, =,$ or $> b$.)

7. (a) $\rho = 2a\tan\theta\sin\theta$; (b) $\rho = 2a\cot\theta\cos\theta$. (Cissoids.)

8. (a) $\rho = a\sin 2\theta$; (b) $\rho = a\cos 2\theta$. (Four-leafed roses.)

9. (a) $\rho = a\sin 3\theta$; (b) $\rho = a\cos 3\theta$. (Three-leafed roses.)

10. (a) $\rho = a\sin 5\theta$; (b) $\rho = a\cos 5\theta$. (Five-leafed roses.)

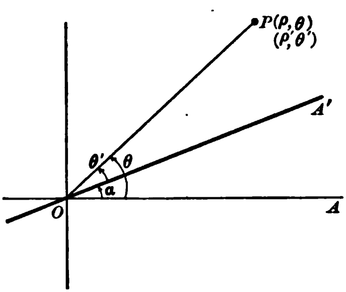
11. $\rho^2 = a^2\sin 2\theta$. (Lemniscate.)

12.* $\rho = a\theta$. (Spiral of Archimedes.)

13.* $\rho = e^{a\theta}$. (Equiangular spiral.)

14.* $\rho^2\theta = a$. (Lituus.)

106. Rotation of Axes. — Rotation of the polar axis does not affect the value of ρ. The figure shows that if the coördinates of P with reference to OA are (ρ, θ), and to OA' are (ρ, θ'), while angle $A'OA$ is α, we have as a formula of rotation



$$\theta = \theta' + \alpha. \qquad (41)$$

By rotating the polar axis the standard equations of curves may be written in a variety of forms.

Translation of the pole is seldom necessary and is best performed by turning the equation into rectangular coördinates.

* Express θ in radians.

107. Symmetrical Transformations. — Equations may also be transformed by the substitutions for symmetry. As in the case of rectangular coördinates it can be shown that the effect of such a substitution is to transform the locus into a curve symmetrical to the given locus with respect to the center or axis involved. Thus we have the following:

If in an equation substitution is made of

$$\left.\begin{array}{r} -\rho \text{ for } \rho \\ -\theta \text{ for } \theta \\ \pi-\theta \text{ for } \theta \end{array}\right\{ \begin{array}{c} \text{the locus of the new equation} \\ \text{is symmetrical to that of the} \\ \text{old with respect to the} \end{array} \left\} \begin{array}{l} \text{pole;} \\ \text{polar axis;} \\ 90^\circ \text{ axis.} \end{array}\right.$$

PROBLEMS

1. In the following find the new equation on rotating the axes through the indicated angle, and draw curve and both axes:

(a) $\rho = a \tan^2\theta \sec\theta,\ \alpha = 90^\circ$; (c) $\rho = a \sin 3\theta,\ \alpha = 30^\circ$;
(b) $\rho = a \sec\theta \pm b,\ \alpha = 90^\circ$; (d) $\rho = a\cos\theta,\ \alpha = 90^\circ$;
(e) $\rho = a + b\cos\theta,\ \alpha = 90^\circ$.

Ans. (a) $\rho = -a\cot^2\theta\csc\theta$.

2. Show that the first of each of the following pairs of equations may be transformed into the second by a proper rotation of the polar axis:

(a) $\rho = a(1+\cos\theta)$, $\rho = a(1-\sin\theta)$;
(b) $\rho = a\sin 2\theta$, $\rho = a\cos 2\theta$;
(c) $\rho = a\sin 3\theta$, $\rho = a\cos 3\theta$;
(d) $\rho^2 = a^2\sin 2\theta$, $\rho^2 = a^2\cos 2\theta$.

3. Find the new equation when the substitution for symmetry with respect to the 90° axis is made, and draw both curves on the same axes:

(a) $\rho = a(1-\cos\theta)$; (c) $\rho = a\tan^2\theta\sec\theta$; (e) $\rho = a\cos 3\theta$;
(b) $\rho = a + b\cos\theta$; (d) $\rho = 2a\tan\theta\sec\theta$; (f) $\rho = a\cos\theta$.

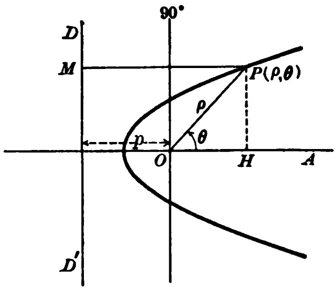
Ans. (a) $\rho = a(1+\cos\theta)$.

4. To each equation apply the substitution for symmetry with respect to the polar axis and draw both curves on the same axes.

(a) $\rho = a(1-\sin\theta)$; (c) $\rho = a\sin 3\theta$;
(b) $\rho = a - b\sin\theta$; (d) $\rho = a\sin\theta$.

Ans. (a) $\rho = a(1+\sin\theta)$.

108. The Polar Equation of the Conic. — If the pole is taken at the focus and the directrix is perpendicular to the polar axis as in the figure, we have at once from the definition of a conic, $OP = eMP$, whence



$$\rho = e(p + \rho \cos \theta),$$

which gives

$$\rho = \frac{ep}{1 - e \cos \theta}. \qquad (42)$$

If the directrix is taken parallel to the polar axis, the equation becomes

$$\rho = \frac{ep}{1 - e \sin \theta}. \qquad (42a)$$

The transformations of rotation and symmetry give eight forms of the conic equation in which the focus is at the pole and the directrix perpendicular or parallel to the polar axis. They are:

$$\rho = \frac{\pm ep}{1 \pm e \cos \theta}; \quad \rho = \frac{\pm ep}{1 \pm e \sin \theta}.$$

In the first set the principal axis is the polar axis; in the second it is the 90° axis. To sketch a conic it is sufficient to put the equation in standard form and find the intercepts. The standard form determines the principal axis and eccentricity, and the intercepts give the vertices and the ends of the latus rectum.

By using the polar equation many of the properties of the conic may be derived more easily than by the use of the rectangular equation. Some of these are indicated in the following problems.

PROBLEMS

1. Transform the equation of the conic, $x^2 + y^2 = e^2(x + p)^2$, into the polar form.

2. Show that the conic is symmetrical with respect to the principal axis. Does the polar form show any other axis of symmetry?

3. Find the latus rectum of the conic. *Ans.* $2\,ep$.

4. Reduce to the standard polar form each of the following conics, find e and p, and draw the curve and its directrix:

(a) $\rho = \dfrac{16}{2 - \cos\theta}$; (e) $\rho = \dfrac{5}{1 + \cos\theta}$;

(b) $\rho = \dfrac{4}{3 + \sin\theta}$; (f) $\rho = \dfrac{6}{1 - \sin\theta}$;

(c) $\rho = \dfrac{8}{1 - 2\sin\theta}$; (g) $\rho = \dfrac{12}{4 + \cos\theta}$;

(d) $\rho = \dfrac{2}{2 + 3\cos\theta}$; (h) $\rho = \dfrac{6}{1 - 2\cos\theta}$;

Ans. (a) $\rho = \dfrac{8}{1 - \frac{1}{2}\cos\theta}$, $e = \frac{1}{2}$, $p = 16$.

5. Show that if $\rho = \dfrac{ep}{1 - e\cos\theta}$ is the equation of a hyperbola, the inclinations of the asymptotes are given by $\cos\theta = \pm\dfrac{1}{e}$.

6. In the hyperbolas of Problem 4, draw the asymptotes.

7. Find the polar intercepts of the conic. From these show that

(a) the major axis of the ellipse is $\dfrac{2\,ep}{1 - e^2}$;

(b) the transverse axis of the hyperbola is $\dfrac{2\,ep}{e^2 - 1}$.

8. Show that the conic never crosses the directrix.

Hint. — The equation of the directrix is $\rho\cos\theta = -p$. Points of intersection are found by solving simultaneously.

9. Derive the equation of the conic when the equation of the directrix is:

(a) $\rho\cos\theta = p$; (b) $\rho\sin\theta = p$.

10. Draw the conic with its directrix for each form:

(a) $\rho = \dfrac{ep}{1 + e\cos\theta}$; (b) $\rho = \dfrac{ep}{1 + e\sin\theta}$;

$\rho = -\dfrac{ep}{1 + e\cos\theta}$; $\rho = -\dfrac{ep}{1 + e\sin\theta}$;

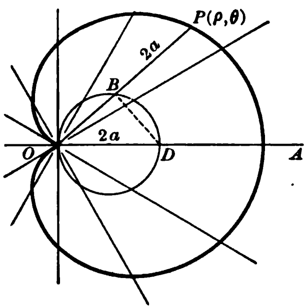
$\rho = \dfrac{ep}{1 - e\cos\theta}$; $\rho = \dfrac{ep}{1 - e\sin\theta}$;

$\rho = -\dfrac{ep}{1 - e\cos\theta}$; $\rho = -\dfrac{ep}{1 - e\sin\theta}$;

11. Show that the locus of $\rho = a\sec^2\dfrac{\theta}{2}$ is a parabola.

109. Equations derived in Polar Coördinates. — The equation of a locus may sometimes be derived more readily in polar than in rectangular coördinates. This is especially true when the locus is described by the end of a line of varying length revolving around a fixed point. The process is substantially the same as when rectangular coördinates are used. The pole should be chosen at a convenient place, usually a fixed point about which some line in the problem revolves. Then a point (ρ, θ) satisfying the conditions is chosen, and relations between ρ, θ and the given arbitrary constants are obtained by means (usually) of trigonometric formulas.

Example. — A chord through the end of a diameter of a circle is extended the length of the diameter. Find the locus of the end of this line.



Choose one end of the given diameter as the pole, and the diameter as the polar axis. Let $2a$ be the diameter of the given circle.

Then from the figure:

$$OB = 2a\cos\theta,\ \rho = OB + 2a.$$

Therefore the equation is

$$\rho = 2a(1 + \cos\theta),$$

and the required locus is a cardioid.

PROBLEMS

1. Find the locus under the conditions of the above problem when the chord is extended a distance $2b$. *Ans.* The limaçon.

2. The radius of a circle whose center is the origin is prolonged a distance equal to the ordinate of its extremity. Find the locus of the end of this line. *Ans.* The cardioid, $\rho = a(1 + \sin\theta)$.

3. Derive a formula for the distance between two points in terms of their polar coördinates.

Hint. — Use the law of cosines.

4. Derive the polar equation of the circle having the center (ρ_1, θ_1).
Ans. $\rho^2 + \rho_1^2 - 2\rho_1\rho\cos(\theta - \theta_1) = r^2$.

5. Derive the polar equation of the ellipse when the pole is at the center. *Ans.* $\rho^2(1 - e^2\cos^2\theta) = b^2$.

Hint. — Use the law of cosines and the fact that the sum of the focal radii is $2a$.

6. Find the locus of the vertex C of a triangle whose base AB is fixed and which has the angle $A = 2C$.
Ans. $\rho = a(2\cos\theta + 1)$, the trisectrix.*

Hint. — Use the law of sines.

7. A fixed point A is at a distance a from a fixed line BC. From A a line is drawn cutting BC in D and points P and P' are chosen on this line b units from D on either side. Find the locus of P and P' as D slides along BC. *Ans.* $\rho = a\sec\theta \pm b$.

8. A tangent is drawn to a circle whose center is the origin and terminated by the x- and y-axis. Find the locus of its mid-point.
Ans. $\rho\sin 2\theta = a$.

9. Find the locus of the mid-points of chords drawn from the end of a fixed diameter of a circle.

10. A line of length $2a$ has its extremities on two fixed perpendicular lines. Find the locus of the foot of a perpendicular from the intersection of the fixed lines to the line of constant length.
Ans. $\rho = a\sin 2\theta$.

11. Given the circle $\rho = 2a\cos\theta$ and the line $\rho\cos\theta = 2a$. From the pole a chord OB is drawn, meeting the line in C. Find the locus of a point P on the line OC, if OP always equals BC.
Ans. $\rho = 2a\tan\theta\sin\theta$, the cissoid.

12. The base of a triangle is in length $2a$. Find the locus of the vertex if the product of the sides is equal to a^2.
Ans. $\rho^2 = 2a^2\cos 2\theta$, the lemniscate.

Hint. — Take the pole at the center of the base and use the law of cosines.

* Since the vertical angle C is one third of the exterior angle at B, this form of the limaçon can be used to trisect any angle, and so is called the trisectrix.

REVIEW PROBLEMS

Identify and sketch the following curves:

1. $\rho = 6 \sin \theta$.
2. $\rho = \dfrac{3}{2 - \sin \theta}$.
3. $\rho = 6 + 4 \sin \theta$.
4. $\rho = 3\theta$.
5. $\rho = 10 \sin 2\theta$.
6. $\rho = 6 \tan \theta \sin \theta$.
7. $\rho = 4 + 6 \sec \theta$.
8. $\rho = \dfrac{3}{1 + 2 \cos \theta}$.
9. $\rho = 8 \cos 2\theta$.
10. $\rho = 8 - 10 \sin \theta$.
11. $\rho \sin \theta = 10$.
12. $\rho = 4 \sec^2 \dfrac{\theta}{2}$.
13. $\rho = 6 \sin 5\theta$.
14. $\rho \cos \theta = -5$.
15. $\rho = \dfrac{5}{1 + \cos \theta}$.
16. $\rho = 5 - 10 \cos \theta$.
17. $\rho\theta = 8\pi$.
18. $\rho^2 = 4 \cos 2\theta$.
19. $\rho = 8 \cos 3\theta$.
20. $\rho = 1 + \sec \theta$.
21. $\rho^2 = 4 \sin^2 2\theta$.
22. $\rho = -2 \cot \theta \csc \theta$.
23. $2\rho - \rho \cos \theta = 8$.
24. $\tan \theta = 1$.
25. $\rho = 10 \sin 3\theta$.
26. $\rho = 8 \tan^2 \theta \sec \theta$.

CHAPTER IX

HIGHER PLANE CURVES

110. Algebraic and Transcendental Equations. — The equations in Cartesian coördinates which we have hitherto treated have been *algebraic* equations, *i.e.* have involved only integral and fractional powers of x and y.

Any equation which is not algebraic (*e.g.* $y + \sin x = 0$) is called *transcendental*, and functions defined by such equations are called transcendental functions. The elementary equations of this class are those in which the exponential, logarithmic, trigonometric, and inverse trigonometric functions are used.

In this chapter we shall discuss a number of curves defined by transcendental equations and algebraic equations of degree higher than the second. Such curves are called *higher plane curves.*

111. The Exponential Curve. — This is defined by the exponential equation

$$y = b^x,$$

where b is any *positive* constant. The quantity b is called the *base.* If the exponent is fractional and involves even roots of b, only the positive value of the root is used.

Discussion. — For simplicity consider the case that $b > 1$.

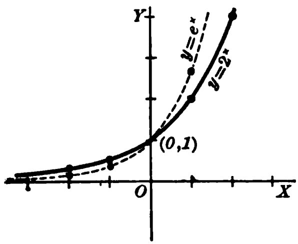
When $x = 0$, $y = b^0 = 1$. If $y = 0$, we would have $0 = b^x$, which is impossible for any value of x. Therefore the curve crosses the y-axis one unit above the origin and does not cross the x-axis.

For all values of x, positive and negative, y is positive, since any power of a positive number is positive. When

$x > 0$, b^x increases with x, since $b > 1$. Therefore in the first quadrant the curve recedes to infinity away from both the x- and y-axes. Since $b^{-x} = \frac{1}{b^x}$, when x approaches $-\infty$, y approaches 0 as a limit. Thus in the second quadrant the curve approaches the x-axis as an asymptote. In the following figure the curve is plotted for two values of b.

The Base e. — The most important case of the exponential is for the base $e\,(= 2.718^+)$, the base of the natural system of logarithms.

In the figure the dotted line is the curve of $y = e^x$ and the heavy line that of $y = 2^x$.



x	$y = e^x$	$y = 2^x$
-2	0.135	0.250
-1	0.368	0.500
0	1.	1.
1	2.718	2.
2	7.389	4.
3	20.086	8.

Exercise 1. Discuss the effect upon the graph of increasing b.

Exercise 2. Discuss the exponential equation for: (*a*) $b < 1$; (*b*) $b = 1$.

112. The Logarithmic Curve. — This is the graph of the equation

$$y = log_b\, x.$$

The base b is always positive and different from 1, and usually greater than 1.

Since in Algebra the logarithm of a number to a given base is defined as that power of the base which makes the result equal to the given number, the equation $y = log_b x$ may be written in the form $x = b^y$. This is the exponential equation with the variables interchanged. Thus the logarithmic

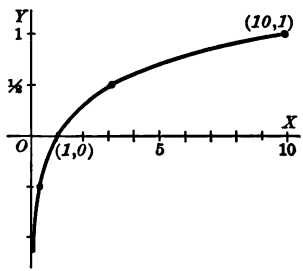
equation is obtained from the exponential $y = b^x$ by solving for x and interchanging variables. Logarithmic and exponential functions are said to be the inverse of each other.

The discussion of the logarithmic equation $y = \log_b x$ will therefore follow from that of the exponential equation $y = b^x$ on interchanging variables. Thus the logarithmic curve crosses the x-axis at (1, 0) and does not cross the y-axis. For $x > 1$, $y > 0$ and as $x \doteq$*∞, $y \doteq \infty$; for $x < 1$, $y < 0$ and as $x \doteq 0$, $y \doteq -\infty$. Since $x = b^y$, the abscissa x is positive for all real values of y, whether positive or negative. This is the reason for the statement that "logarithms of negative numbers do not exist."

The following graph is the logarithmic curve for $b = 10$. The table of values may be computed by using a table of logarithms or by writing the equation in the form $x = 10^y$ and computing values of x corresponding to various values of y. Note that the logarithmic curve is symmetrical to the exponential curve with respect to the line $y = x$.

$y = \log_{10} x$

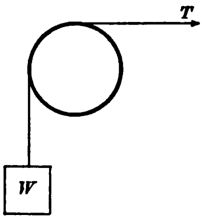
x	y	x	y
1.0	0	.1	-1
3.1	.5	.01	-2
10	1	.001	-3
100	2	.0001	-4
etc.	etc.	etc.	etc.



113. Applications. — The exponential and logarithmic curves show clearly the method of variation of these functions. Thus for $b > 1$, the exponential function increases with great rapidity as x approaches ∞, while the logarithmic function increases very slowly.

* This symbol is used for the words "approaches as a limit."

These functions not only have a theoretical importance, but constantly appear in the statements of the laws of Physics and other sciences. For example, the adjoining figure represents a weight W suspended by a rope wrapped several times about a wooden beam and kept from falling by a tension T. The relation between W and T is given by the equation



$$W = Te^{\mu x},$$

where x represents the number of times the rope is wound around the beam and μ is a constant depending on the friction between the rope and the beam. For a hemp rope on smooth oak $\mu = 3.34$, nearly.

PROBLEMS

1. Plot the system of curves defined by $y = b^x$ for $b = 1, 2, 3$, and 10; all on the same axes.

2. Plot the system defined by $y = b^x$ for $b = 1, \frac{1}{2}$ and $\frac{1}{4}$; all on the same axes.

3. Plot the logarithmic curve for the bases 10, 3, e, and 2; all on the same axes.

4. Plot the graphs of:

(a) $y = 2^{-x}$; (c) $y = x^x$; (e) $y = e^{-x}$;

(b) $y = 4^{-x}$; (d) $y = 6^{\frac{1}{x}}$; (f) $y = e^{2x}$.

5. Plot the graphs of:

(a) $y = \log_2 x$; (c) $2x = \log_{10} y$;

(b) $y = 3 \log_2 x$; (d) $x = \log_{10} 2^y$.

6. Write each of the equations of Problem 4 in logarithmic form.

Ans. (a) $x = -\log_2 y$.

7. Write each of the equations of Problem 5 in exponential form.

Ans. (a) $x = 2^y$.

8. Discuss the following equations for symmetry and plot their graphs:

(a) $y = \dfrac{e^x + e^{-x}}{2}$. This is called the *hyperbolic cosine* and written

$$y = \cosh x.$$

(b) $y = \frac{e^x - e^{-x}}{2}$, or $y = \sinh x$.

(c) $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$, the equation of the *catenary.**

9. Show that the graphs of $y = b^x$ and $y = \left(\frac{1}{b}\right)^x$ are symmetrical with respect to the y-axis.

10. Plot the curve $y = 3^x$ and estimate the logarithms to the base 3 of the integers from 2 to 10.

11. Plot the graph of the function $W = Te^{\mu x}$, where $T = 100$ lbs. and $\mu = 1.1$. Estimate from the graph how many turns of the rope would be required for this tension to support 1 ton. 2 tons. 10 tons.

Hint. — This makes $e^{\mu} = 3$, nearly.

114. Periodic Functions. — *A periodic function of a variable is a function whose values are repeated at definite intervals as the variable increases.* Cosine x is an example. It passes successively through all values from $+1$ to -1 and back to $+1$ as its angle increases from 0 to 2π, and repeats these values in the same order as often as the angle x increases by 2π; or n times for a change of $2n\pi$ in the angle.

The interval of repetition is called the *period* of the function. If k is the period of $f(x)$, then $f(x+k) = f(x)$. The maximum numerical value of the function is called the *amplitude.* The table illustrates these definitions for several functions.

$f(x)$	Period	Amplitude
Sin x	2π	1
Cos x	2π	1
Tan x	π	∞
Sin $2x$	π	1
Cos $\frac{x}{n}$	$2n\pi$	1

115. The Sine Curve. — This is plotted from the equation $y = \sin x$, where x is to be reckoned in circular, or radian measure.

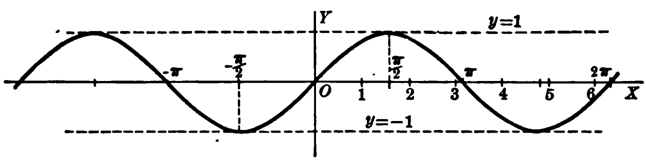
* This is the form assumed by a perfectly flexible cord or chain suspended between two points.

We may assume values for x at any convenient interval, as $\frac{\pi}{6}$, and take the corresponding values of sin x from a table of natural sines.

x (radians)	x (degrees)	$\sin x$	$\cos x$
0	0	.00	1.00
$\frac{\pi}{6} = .52$	30	.50	.87
1.05	60	.87	.50
$\frac{\pi}{2} = 1.57$	90	1.00	.00
2.09	120	.87	$-.50$
2.62	150	.50	$-.87$
$\pi = 3.14$	180	.00	-1.00
3.66	210	$-.50$	$-.87$
4.19	240	$-.87$	$-.50$
4.71	270	-1.00	.00
5.24	300	$-.87$	.50
5.76	330	$-.50$	.87
$2\pi = 6.28$	360	.00	1.00
6.81	390	.50	.87
7.33	420	.87	.50
etc.	etc.	etc.	etc.

In the accompanying table the values of x in degrees are found in column 2 and their circular measures to be used in plotting are in column 1. The circular measure of x is the abscissa and the value of its sine is the ordinate.

In plotting it is convenient to lay off on the x-axis $\frac{\pi}{6} = .52$ units and, using this as a unit, to mark off the various abscissas. After the values from the table up to 2π have been plotted the curve is constructed through one period, and the



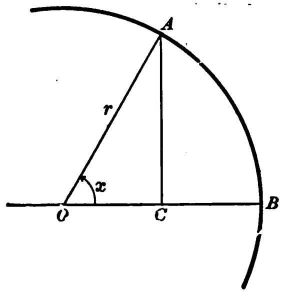
same values of y may be used to construct it through successive periods.

Since the angle x may be taken negative as well as positive, the curve may be extended indefinitely in the negative direction. If we substitute $(-x, -y)$ for (x, y) in the equation, we have

$$-y = \sin(-x) = -\sin x, \text{ or } y = \sin x,$$

and the form of the equation is unchanged. Hence the curve is symmetrical with respect to the origin.

116. Circular Measure. — For plotting the sine curve x can be measured in degrees, and any convenient unit taken as a degree, but the scale of measurement in x will not correspond to that in y unless circular measure is used. Circular measure gives the same scale for both coördinates. For in radians,



$$\text{angle} = \frac{\text{arc}}{\text{radius}},$$

or $$x = \frac{AB}{r}.$$

But $$y = \sin x = \frac{AC}{r}.$$

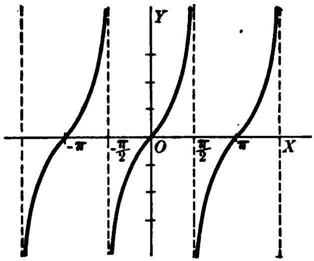
So as the angle x is generated by rotating the side OA from the initial position OB, the lengths of AB and AC *in terms of the radius as a unit* are the values of x and $\sin x$ given in columns 1 and 3 of the table of values for plotting. This is only true where x is expressed in circular measure.

117. The Cosine Curve. — The curve of cosines is plotted from the equation $y = \cos x$. This curve has the same period and amplitude as the sine curve, and in fact has the same shape; but it passes through (0, 1) instead of the origin.

This is true because the cosine of any angle is equal to the sine of an angle 90° greater, or $\cos x = \sin (90° + x)$. Hence the table of sines can be used for plotting the cosine curve by moving the whole column of sine values up 90°, so that the sine curve becomes the cosine curve when the origin is advanced to the point $\left(\frac{\pi}{2}, 0\right)$.

118. The Tangent Curve. — The curve of tangents is plotted from the equation $y = \tan x$. It passes through the origin and its period is π. The value of y is infinite when x is an odd multiple of $\frac{\pi}{2}$, and the curve passes through all values of y between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$, and so on at intervals of π, the period.

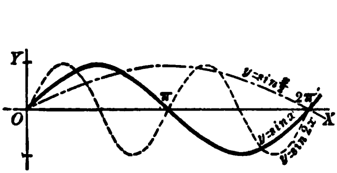
The table of values follows.

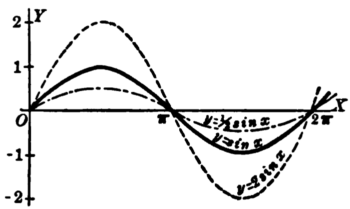


x	$\tan x$
$-\frac{\pi}{2}$	$-\infty$
$-\frac{\pi}{3}$	-1.73
$-\frac{\pi}{6}$	$-.58$
0	0.
$\frac{\pi}{6}$	.58
$\frac{\pi}{3}$	1.73
$\frac{\pi}{2}$	∞

119. Multiple Angles. — Consider $y = \sin nx$. *The multiple n divides the period of the function by n, but does not alter the amplitude.* To prove this, substitute $x + \frac{2\pi}{n}$ for x. We have

$$\sin n\left(x + \frac{2\pi}{n}\right) = \sin\,(nx + 2\pi) = \sin nx.$$





Hence $\frac{2\pi}{n}$ is the period of $\sin nx$. That the amplitude is unaltered is obvious.

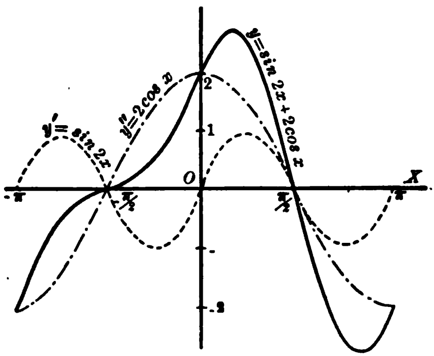
Compare this with $y = n \sin x$, where the period is unaltered, but the amplitude is multiplied by n. All of this appears clearly in the plots. Thus we have

for $y = \sin x$, the period is 2π and $y = 1$, when $x = 90°$;

for $y = \sin 2x$, the period is π and $y = 1$, when $x = 45°$;

for $y = 2 \sin x$, the period is 2π and $y = 2$, when $x = 90°$.

120. Sum of Functions. — Sometimes it is necessary to plot functions which are the sum of two or more trigonometric functions. Time is saved in making the table of values by writing the values of the various terms in parallel columns and summing corresponding values to get the

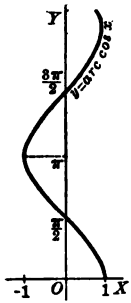


various ordinates. Thus to plot $y = \sin 2x + 2 \cos x$, make a table of values for $\sin 2x$ and $2 \cos x$ in parallel columns and write the sum of values corresponding to the same value of x in a third column for the ordinate to be plotted.

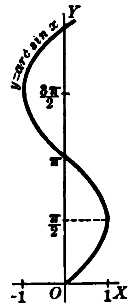
An alternative method is to plot the curves $y' = \sin 2x$ and $y'' = 2 \cos x$ on the same axes and obtain the corresponding ordinates by measurement. The figure illustrates this method.

121. Inverse Trigonometric Functions. — The relation expressed in the inverse notation by $y = \sin^{-1} x$ reads in direct notation $x = \sin y$.

$y = \cos^{-1} x = \text{arc} \cos x.$



$y = \sin^{-1} x = \text{arc} \sin x.$



The curve of inverse sines is therefore the same as the curve of sines in shape, period, and amplitude, but the axes are interchanged.

The notation arc sin x is preferable to $\sin^{-1} x$.

PROBLEMS

1. Discuss and plot the graph of:

(a) $y = \cos x$;
(b) $y = \cot x$;
(c) $y = \sec x$;
(d) $y = \csc x$.

2. Find the period and amplitude and draw the graph of:

(a) $y = \sin 3x$;
(b) $y = \cos 5x$;
(c) $y = \tan \frac{x}{2}$.
(d) $y = 3 \sec 3x$;
(e) $y = \frac{3}{2} \cos \frac{2}{3} x$.

3. Plot the graph of:

(a) $y = \text{arc} \tan x$;
(b) $y = \text{arc} \sec x$;
(c) $y = 2 \,\text{arc} \cos 2x$:
(d) $y = 3 \,\text{arc} \sin \frac{x}{3}$.

4. Draw on the same axes the graphs of:

(a) $\begin{cases} y = \sin x, \\ y = \csc x; \end{cases}$ (b) $\begin{cases} y = \cos x, \\ y = \sec x; \end{cases}$ (c) $\begin{cases} y = \tan x, \\ y = \cot x. \end{cases}$

What relation holds between corresponding ordinates of each pair of curves?

5. Draw each of the following curves. For any value of x, draw ordinates at $\pm x$, $\pi \pm x$, $2\pi \pm x$, and then write the trigonometric formulas suggested by the figure.

(a) $y = \sin x$; (c) $y = \tan x$; (e) $y = \sec x$;
(b) $y = \cos x$; (d) $y = \cot x$; (f) $y = \csc x$.

Ans. (a) $\sin(-x) = -\sin x$;
$\sin(\pi \pm x) = \mp \sin x$;
$\sin(2\pi \pm x) = \pm \sin x$.

6. Plot the graphs of the following equations:

(a) $y = \cos 2x - \sin \frac{x}{2}$; (c) $y = x + 2\sin x$;

(b) $y = \sec x - x$; (d) $y = \cos \frac{x}{2} + \sin 2x$.

7. Determine which of the functions of Problem 6 are periodic and find the periods.

Ans. (a) and (d) have the period 4π; (b) and (c) are not periodic.

122. Parametric Equations. — In some curves the coördinates are functions of a third variable such that the conditions determining the curve may conveniently be expressed by two equations between these three variables instead of by a single equation in x and y. This third variable is called a *parameter*, and the two equations *parametric equations* of a curve. If t is the parameter, the parametric equations of the curve are usually of the form $x = f(t)$, $y = g(t)$.

The parameter usually represents some geometric magnitude, or the time during which the point tracing the curve has been in motion; but it may be chosen in any manner that is convenient. Thus for the semi-cubical parabola $y^2 = ax^3$, we may assume a parameter t and write $x = at^2$, $y = a^2t^3$, since by eliminating t between these equations we obtain the original form.

To plot the graph of such equations we take various values of the parameter and compute the corresponding values of x and y. Then plot the points (x, y) as usual. The values of the parameter appear only in the table of values, and not in locating the point.

123. Parametric Equations of the Circle. — As usual we take the origin at the center of the circle. Then the circle is generated by rotating OP. Taking θ as the parameter, we have at once from the figure and the definitions of the sine and cosine:

$$x = r \cos \theta, \qquad y = r \sin \theta.$$

These are the parametric equations of the circle. From the table of values below points on the curve are (10, 0), (8.7, 5), (5, 8.7), (0, 10), (– 5, 8.7), (– 8.7, 5), (– 10, 0), (– 8.7, – 5), etc.

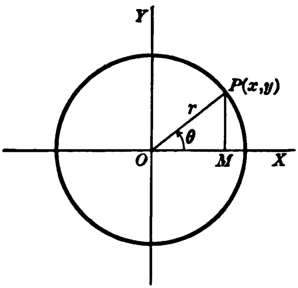
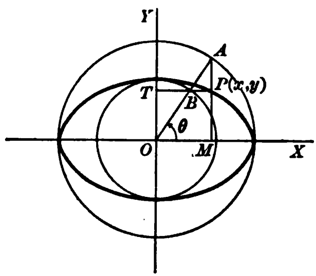


TABLE OF VALUES FOR $r = 10$

θ	x	y
0	10	0
30	8.7	5.
60	5.	8.7
90	0	10
120	– 5.	8.7
150	– 8.7	5.
180	– 10.	0.
210	– 8.7	– 5

Note that if we eliminate θ between the parametric equations, we get the standard form $x^2 + y^2 = r^2$.

124. Parametric Equations of the Ellipse. — Suppose we have a circle of radius a and center O, in which OA is any radius and AM a perpendicular from A upon a fixed diameter. B is a point on the radius at a distance b from O and P is the projection of B on AM. It is required to find the locus of P



Take the fixed diameter as the x-axis and the center as the

origin. Let (x, y) be the coördinates of P and $\theta = \angle MOA$ be the parameter. Then we have at once from the figure,

$$x = OM = OA \cos\theta = a\cos\theta,$$
$$y = MP = OB \sin\theta = b\sin\theta.$$

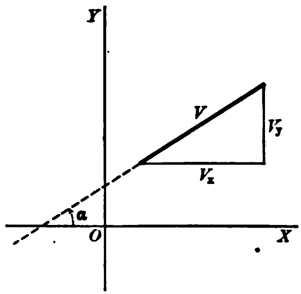
These are the parametric equations of the locus. It remains to show that the curve is an ellipse. Solving the equations for $\cos\theta$ and $\sin\theta$ respectively, squaring and adding we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta = 1,$$

the equation of the ellipse.

The circles described by B and A are called the *auxiliary circles* of the ellipse. By drawing them on coördinate paper, the various positions of P can be readily found and any ellipse of given semi-axes plotted with accuracy.

125. Path of a Projectile. — If an object moves with a velocity of V feet per second along a line inclined to the x-axis at an angle α, it is clear from the adjoining figure that at the end of each second its distance from the y-axis has been increased by $V_x = V\cos\alpha$ feet, and that from the x-axis by $V_y = V\sin\alpha$ feet. These quantities, V_x and V_y, are called the x- and *y-components*, respectively, of the velocity, and when multiplied by the time give the total displacement parallel to the x- and y-axes.



In the case when the motion is in a vertical plane, the vertical velocity does not remain constant, but is affected by the force of gravity. In this case we find in Mechanics that the height is given by the formula $h = vt - 16t^2$, where

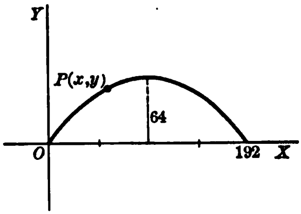
h is the height in *feet*, t the time in *seconds*, and v the vertical component of the initial velocity in *feet per second.**

Using these principles, let us solve the problem: Find the path of a projectile discharged at an angle $\alpha = \text{arc}\cos\frac{3}{5}$ with an initial velocity of 80 feet per second.

Taking the point of projection as the origin, the y-axis vertical to the earth's surface and the x-axis horizontal in the direction of projection, let $P(x, y)$ be the position of the projectile after t seconds.

The horizontal component of the initial velocity is $V_x = 80\cos\alpha = 80\cdot\frac{3}{5} = 48$; the vertical component is $V_y = 80\sin\alpha = 80\cdot\frac{4}{5} = 64$. As the horizontal velocity remains constant,

$$x = V_x t = 48t. \qquad (a)$$



Using the above formula,

$$y = V_y t - 16t^2 = 64t - 16t^2 \qquad (b)$$

Equations (a) and (b) are the parametric equations of the locus. To identify the curve, eliminate t and we have

$$y = \tfrac{4}{3}x - \tfrac{1}{144}x^2,$$

evidently a parabola.

To get the *range*, set $y = 0$, and we have $x = 192$ feet. If we complete the square, we have

$$(x - 96)^2 = -144(y - 64).$$

Hence the vertex is (96, 64), *i.e.* the ***greatest height*** reached is 64 feet.

PROBLEMS

1. Plot the graphs of the following parametric equations:

(a) $x = 3t$, $y = \dfrac{1}{6t}$; (e) $x = t + 1$, $y = \dfrac{t^3}{2}$;

(b) $x = t^2$, $y = 2t$; (f) $x = a\cos\theta$, $y = b\sin^3\theta$;

(c) $x = 5\cos\theta$, $y = 3\sin\theta$; (g) $x\cos\theta = a$, $y = a\tan\theta$;

(d) $x = a\cos^3\theta$, $y = b\sin^3\theta$; (h) $x = 2 + 3\cos\theta$, $y = 3\sin\theta - 1$.

2. Eliminate the parameter in each pair of equations in Problem 1.

3. Form parametric equations for the folium of Descartes $x^3 + y^3 = 3axy$, make a table of values and plot the curve.

Hint. — Set $y = tx$.

Ans. $x = \dfrac{3at}{1 + t^3}$, $y = \dfrac{3at^2}{1 + t^3}$.

* In this and all that follows, we disregard air resistance. The results are therefore first approximations to the true ones.

4. A straight line has the x-intercept 5 and the inclination 30°. Find its parametric equations when the parameter t is the distance of the tracing point from the intersection with the x-axis.

Ans. $x = 5 + t\frac{\sqrt{3}}{2}, \ y = \frac{t}{2}.$

5. Find the locus of the projectile, the range, and highest point reached from the following data:

(*a*) initial velocity 576 feet per second, angle of projection 45°;
(*b*) initial velocity 400 feet per second, angle of projection 30°;
(*c*) initial velocity 800 feet per second, angle of projection 45°.

6. In Problem 5 (*a*), the projectile is fired up a hill of slope $\frac{1}{2}$. Find how far up the hill it will strike the ground.

Ans. 5796 feet.

7. Find the equation of the locus of a projectile of initial velocity V and angle of projection α. *Ans.* $y = x \tan \alpha - \frac{16 x^2}{V^2 \cos^2 \alpha}.$

8. In Problem 7 find the range and greatest height.

Ans. Range $\frac{V^2 \sin 2\alpha}{32}$; greatest height $\frac{V^2 \sin^2 \alpha}{64}.$

9. A fly crawls on a spoke of a wheel from the hub toward the rim at a rate of 4 feet per minute. The wheel makes 2 revolutions per minute. Find the equation of the path described by the fly.

Ans. (in polar coördinates) $\rho = \frac{\theta}{\pi}.$

10. A point moves around a circle of radius 10 feet 50 times per minute. Write the parametric equations of the circle, using the time during which the point has been in motion as a parameter.

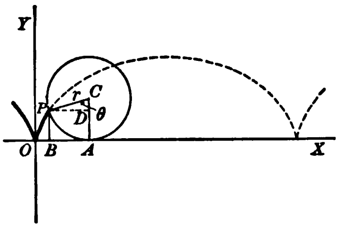
11. The parametric equations of the path of a point are

$$x = a \cos kt,$$
$$y = 0,$$

where x is measured in feet, t is measured in seconds and a and k are constants. Plot the locus. How often does the point pass through the origin?

126. The Cycloid. — If a circle rolls along a straight line the curve traced by a point on the circumference is called a *cycloid*. The straight line is called the base.

As the circle rolls it describes the length of the circumference on the base at each revolution, and a proportional distance for any part of a revolution. Hence the point of the circle touching the base is always at a distance from the starting point equal to the length of the arc subtending θ.



In the figure the tracing point P starts from the point of contact O, which is taken as the origin, and the base is taken as the x-axis.

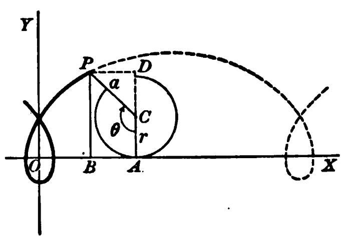
Call the angle which the radius to the tracing point makes with the vertical, θ. Then $OA = PA = r\theta$, since arc $=$ angle $\times$ radius. This gives

$$x = OA - BA = r\theta - r\sin\theta,$$

$$y = AC - DC = r - r\cos\theta,$$

which are the parametric equations of the cycloid.

The Prolate and Curtate Cycloids. If the tracing point is taken anywhere on the radius of the rolling circle or the radius produced, the curve is called the *prolate cycloid* when P is without the circle and the *curtate cycloid* when P is within. The term *trochoid* is also used for both prolate and curtate cycloids.



Let a be the distance from the center to the tracing point. The parametic equations of both prolate and curtate cycloids are

$$x = r\theta - a\sin\theta,$$

$$y = r - a\cos\theta.$$

They are derived in precisely the same manner as those of the ordinary cycloid.

PROBLEMS

1. Sketch the path of a point on a 28-inch bicycle wheel for two revolutions. Find the base of the cycloid and the coördinates of the tracing point when the wheel has made one third, one half and two thirds of a revolution.

2. Plot the cycloid for $r = 10$, forming a table of values for θ, x, and y.

3. Eliminate θ between the parametric equations of the cycloid.

Hint. — $\theta = \text{arc cos}\left(\frac{r-y}{r}\right)$ and $\text{vers}\,\theta = 1 - \cos\theta$,

Ans. $x = r\,\text{arc vers}\,\frac{y}{r} - \sqrt{2ry - y^2}$.

4. Write out in full the derivation of the parametric equations of (*a*) the curtate cycloid; (*b*) the prolate cycloid.

5. Trace the curtate cycloid though two periods when $r = 6$ and $a = 5$, and find the coördinates of the tracing point when $\theta = 120°$.

6. Plot the prolate cycloid for $r = 10$ and $a = 12$.

7. Show from the equations of the cycloid that y is a periodic function of x, of period $2\pi r$.

8. Construct the figure and derive the equations of the cycloid when $\theta > 90°$.

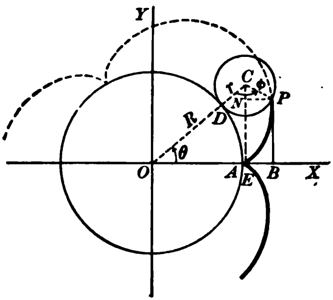
9. Show that the first arch of the cycloid is symmetrical with respect to the line $x = \pi r$.

Hint. — Show that for any value of y, as y_1, $\theta = \theta_1$, or $2\pi - \theta_1$. From this get two values of x, x_1 and x_2, and show that $x_2 - \pi r = \pi r - x_1$. Then apply the definition of symmetry.

127. The Epicycloid and Hypocycloid. — When the base is a circle instead of a straight line, the curve is called an *epicycloid* if the generating circle rolls on the outside of the base, and a *hypocycloid* if it rolls inside.

Note. — The epicycloid is the basis of the "Theory of Epicycles" which was used in the Ptolemaic system of astronomy to explain the motions of the planets.

Let the tracing point start from A and the generating circle move on the outside until its contact with the base is at D. Thus the arc AP of an epicycloid is described and the arc PD has rolled over the arc AD of the base.



Let the angle which the generating radius CP makes with the line of centers be ϕ, and the angle of this line CO with the x-axis be θ.

Using circular measure, since arc $AD =$ arc PD,

$$R\theta = r\phi. \tag{a}$$

Now CPN is the supplement of $\theta + \phi$,

and $\therefore \sin CPN = \sin(\theta + \phi)$,

$$\cos CPN = -\cos(\theta + \phi).$$

But
$$x = OB = OE + NP$$
$$= (R + r)\cos\theta - r\cos(\theta + \phi)$$

and
$$y = BP = EC - NC$$
$$= (R + r)\sin\theta - r\sin(\theta + \phi).$$

These equations contain two parameters, θ and ϕ. ϕ may be eliminated by substituting its value from (a), which gives

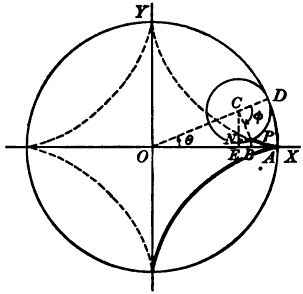
$$x = (R + r)\cos\theta - r\cos\frac{R + r}{r}\theta,$$

$$y = (R + r)\sin\theta - r\sin\frac{R + r}{r}\theta.$$

If r is taken negative we have the equations of the hypocycloid:

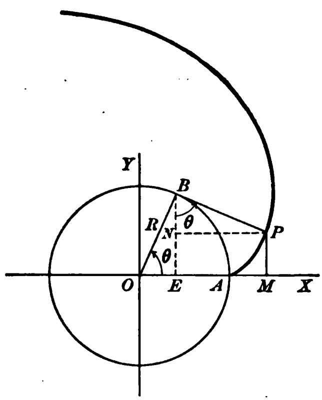
$$x=(R-r)\cos\theta+r\cos\frac{R-r}{r}\theta,$$

$$y=(R-r)\sin\theta-r\sin\frac{R-r}{r}\theta.$$



The Astroid. — In the hypocycloid $R=4r$, the rolling circle makes four revolutions in passing around the base, forming the astroid, or hypocycloid of four cusps.

128. The Involute of the Circle. — A string is wound about the circumference of a circle. One end is fastened to the circumference and the string is unwound. If the string is kept stretched the curve traced by the free end is called the *involute* of the circle.



If it begins to unwind at A, the arc AP of the involute is traced when the string has unwound as far as B, and the part unwound is BP.

BP is tangent to the circle at B, and

$$BP=BA=R\theta.$$

The equations are

$$x=R\cos\theta+R\theta\sin\theta,$$
$$y=R\sin\theta-R\theta\cos\theta.$$

NOTE. — The teeth of large-geared wheels are cut in the shape of epi- or hypocycloids as the best forms to avoid the binding and friction that occur between straight teeth.

PROBLEMS

1. Show that for the astroid the equations reduce to

$$x = R\cos^3\theta,$$
$$y = R\sin^3\theta,$$

whence $$x^{\frac{2}{3}} + y^{\frac{2}{3}} = R^{\frac{2}{3}}.$$

2. Simplify the equations of the epicycloid and hypocycloid, and draw the curves when

$$(a)\ R = r;\ (b)\ R = 2r.$$

3. Show that the epicycloid in which $R = r$ is a cardioid.

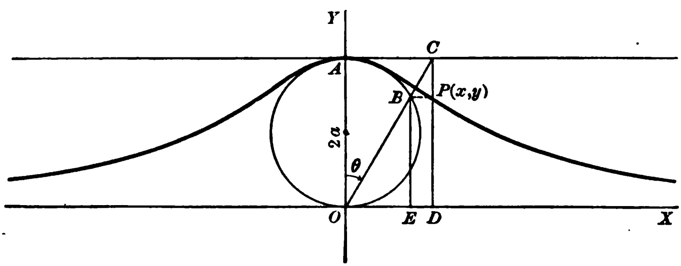
4. Draw the epicycloid and hypocycloid when

$$(a)\ 4R = 3r;\ (b)\ 3R = 5r;\ (c)\ r = 2R.$$

5. Derive the equations of the hypocycloid from a figure.

129. Algebraic Curves. — Algebraic curves of degree higher than the second have a great variety of forms and are best studied with the aid of Calculus. Two simple examples of these curves are the cubical parabola $y = ax^3$ and the semi-cubical parabola $y^2 = ax^3$. The majority of the curves in Chapter VIII are found to be algebraic curves if their equations are transformed into rectangular form. Many of these curves are of historical importance, as the limaçon, the conchoid, the cissoid, and the lemniscate, given in Chapter VIII, and the witch, the strophoid, and the ovals of Cassini, which will be discussed in the following sections.

130. The Witch. — A circle of radius a is drawn tangent to the x-axis through the origin, meeting the y-axis in A.



Through A a tangent is drawn to the circle, and from O a secant is drawn meeting the circle in B and the tangent in C. The locus of the projection of B upon the ordinate of C is called the *witch*.

To find its equation take the angle $\theta = AOC$ as a parameter. Then we have at once

$$x = OD = AC = 2\,a \tan\theta,$$

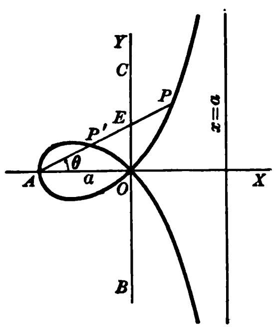
and $$y = DP = EB = OB \cos\theta = 2\,a \cos^2\theta.$$

Eliminating the parameter θ, the equation is

$$y = \frac{8\,a^3}{x^2 + 4\,a^2}.$$

The form of the equation shows that the curve is symmetrical with respect to the y-axis and has the x-axis as an asymptote.

131. The Strophoid. — The distance of a fixed point A from a fixed line BC is a. From A a perpendicular AO is drawn to BC, also another line meeting BC at E. On this last line points P and P' are taken such that $P'E = EP = OE$. The locus of the points P and P' is a curve called the *strophoid*.



To find its locus take the fixed line as the y-axis and the x-axis through the fixed point. Let the coördinates of P and P' be denoted by (x, y). Taking the angle OAE as a parameter θ, we have at once that

$$P'E = EP = OE = a \tan\theta.$$

Hence $$x = \pm\, EP \cos\theta = \pm\, a \sin\theta;$$

and $$y = OE \pm EP \sin\theta = a \tan\theta\,(1 \pm \sin\theta).$$

By eliminating the parameter, we have the equation,

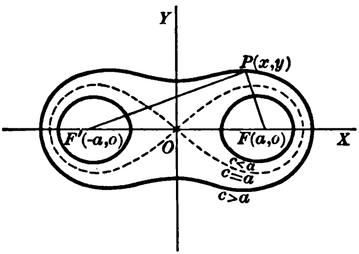
$$y = \pm x\frac{\sqrt{a+x}}{\sqrt{a-x}},$$

or

$$y^2 = x^2\frac{a+x}{a-x}.$$

The curve is evidently symmetrical with respect to the x-axis and has the line $x = a$ as an asymptote.

132. The Ovals of Cassini. — The locus of the vertex of a triangle which has a constant base and has the product of its other two sides equal to a given constant is called the *ovals of Cassini*.



Call the length of the base $2a$, take the origin at its mid-point, and let the x-axis lie along the base. Then the conditions of the problem make $F'P \cdot FP$ equal to a constant, which we call c^2 since it is essentially positive.

Using the distance formula, we have

$$\sqrt{(x+a)^2+y^2}\cdot\sqrt{(x-a)^2+y^2} = c^2.$$

Simplifying, this reduces to

$$(x^2+y^2+a^2)^2 - 4a^2x^2 = c^4.$$

Three types are possible, according to the relative values of c and a. If $c = a$, this last equation may easily be reduced to the form,

$$(x^2+y^2)^2 = 2a^2(x^2-y^2),$$

which becomes on transformation to polar coördinates,

$$\rho^2 = 2a^2\cos 2\theta,$$

showing that in this case the curve is a lemniscate. (See page 134.)

PROBLEMS

1. Transform into rectangular form the polar equations of the following curves:

(*a*) the limaçon;
(*b*) the cardioid;
(*c*) the conchoid;
(*d*) the cissoid;
(*e*) the four-leafed rose.

2. Show that the following curves are not algebraic:

(*a*) the spiral of Archimedes;
(*b*) the lituus.

3. Plot the witch from its parametric equations.

4. Plot the strophoid.

5. Find the polar equation of the strophoid.

Ans. $\rho = -\dfrac{a \cos 2\theta}{\cos \theta}$.

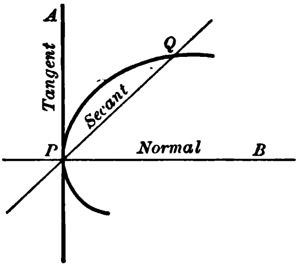
6. Transform the rectangular equation of the ovals of Cassini into polar form.

7. Derive from the figure the polar equation of the ovals of Cassini.

CHAPTER X

TANGENTS AND NORMALS

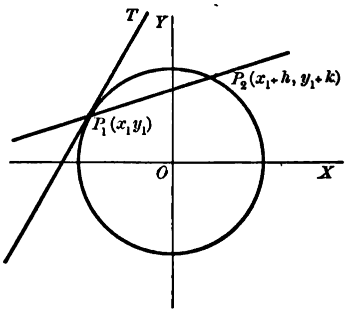
133. Definitions. — A line cutting a curve is called a *secant.* In the figure PQ is a secant. If Q is made to move along the curve towards P, it is clear that PQ will turn about P and approach a limiting position PA, which is called the position of tangency. Therefore



The tangent to a curve at a given point is the limiting position of a secant when two of its intersections approach coincidence at that point.

The normal to a curve at a given point is the line perpendicular to the tangent at that point.

134. Slope of Tangent and Normal to a Circle at a Given Point. — To find the slope of a tangent to a curve at a given point $P_1(x_1, y_1)$, we first find the slope of the secant through P_1 and a neighboring point $P_2(x_1 + h, y_1 + k)$ on the curve. This will be $\frac{k}{h}$. As P_2 is made to approach P_1 as a limit, the secant approaches the tangent and the slope of the tangent is the limiting



value of $\frac{k}{h}$, which we write $\lim \frac{k}{h}$. Since both k and h approach zero, the limiting value of $\frac{k}{h}$ has the form $\frac{0}{0}$, which is called an indeterminate form. To find the value of such a limit special methods must be employed.

In the case of the circle we proceed as follows. Since P_1 and P_2 are both on the circle, their coördinates satisfy the equation of the circle. Hence

$$x_1^2 + y_1^2 = r^2,$$

$$(x_1 + h)^2 + (y_1 + k)^2 = r^2.$$

Expanding and subtracting,

$$2\,hx_1 + h^2 + 2\,ky_1 + k^2 = 0,$$

$$h(2\,x_1 + h) = -\,k(2\,y_1 + k),$$

and

$$\frac{k}{h} = -\frac{2\,x_1 + h}{2\,y_1 + k}.$$

From this last relation, we have

$$\lim \frac{k}{h} = -\lim \frac{2\,x_1 + h}{2\,y_1 + k} = -\frac{x_1}{y_1}.$$

Therefore for the tangent to the circle $x^2 + y^2 = r^2$,

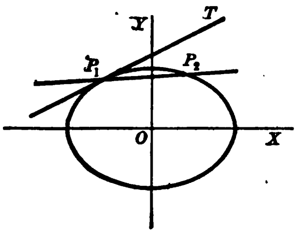
$$m = -\frac{x_1}{y_1}. \tag{43}$$

As the normal is perpendicular to the tangent, its slope is the negative reciprocal of this expression, or $\frac{y_1}{x_1}$.

135. Slope of Tangent and Normal to an Ellipse at a Given Point. — The method of procedure is the same as that for the circle. Let the equation of the ellipse be

$$b^2x^2 + a^2y^2 = a^2b^2,$$

and the point of tangency be $P_1(x_1, y_1)$. The slope of a secant through P_1 and a neighboring point $P_2(x_1+h, y_1+k)$ on the curve is as before $\frac{k}{h}$.



Substituting the coördinates of P_1 and P_2 in the given equation,

$$b^2x_1^2 + a^2y_1^2 = a^2b^2,$$

$$b^2(x_1 + h)^2 + a^2(y_1 + k)^2 = a^2b^2.$$

Expanding and subtracting,

$$2\,b^2x_1h + b^2h^2 + 2\,a^2y_1k + a^2k^2 = 0,$$

$$h(2\,b^2x_1 + b^2h) = -k(2\,a^2y_1 + a^2k),$$

$$\frac{k}{h} = -\frac{2\,b^2x_1 + b^2h}{2\,a^2y_1 + a^2k}.$$

Hence $\lim \frac{k}{h} = -\lim \frac{2\,b^2x_1 + b^2h}{2\,a^2y_1 + a^2k} = -\frac{2\,b^2x_1}{2\,a^2y_1}.$

Thus for the tangent to the ellipse, the slope is

$$m = -\frac{b^2x_1}{a^2y_1}. \qquad (44)$$

The slope of the normal is $\frac{a^2y_1}{b^2x_1}$.

136. Slope of Tangent and Normal to the Parabola and Hyperbola at a Given Point. — By applying the method of §§ 134, 135 to the equations of the parabola and hyperbola, $y^2 = 2\,px$ and $b^2x^2 - a^2y^2 = a^2b^2$, we obtain the following results:

Tangent to parabola: $m = \frac{p}{y_1}$. (45)

Tangent to hyperbola: $m = \frac{b^2x_1}{a^2y_1}$. (46)

The slopes of the normals are: for the parabola, $-\dfrac{y_1}{p}$;

for the hyperbola, $-\dfrac{a^2y_1}{b^2x_1}$.

Exercise 1. Prove formula (45).

Exercise 2. Prove formula (46).

137. Equations of the Tangent and Normal at a Given Point. — To find the equation of the tangent or normal to a curve at a given point, it is necessary only to find the slope and then use the point-slope equation of the straight line,

$$y - y_1 = m(x - x_1).$$

Exercise 3. Show that the equation of the tangent to the circle $x^2 + y^2 = r^2$ at any point $P_1(x_1, y_1)$ reduces to $x_1x + y_1y = r^2$.

Hint. — Since P_1 is on the circle, simplify by using $x_1^2 + y_1^2 = r^2$.

Exercise 4. Show that the equation of the tangent at any point $P_1(x_1, y_1)$ is:

(*a*) for the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $\quad \dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = 1$;

(*b*) for the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $\quad \dfrac{x_1x}{a^2} - \dfrac{y_1y}{b^2} = 1$;

(*c*) for the parabola $y^2 = 2px$, $\quad y_1y = p(x + x_1)$.

PROBLEMS

1. Find the slope of the tangent to each of the following curves for any point (x_1, y_1) on the curve:

(*a*) $xy = 2a^2$;
(*b*) $y = ax^3$;
(*c*) $y^2 = ax^3$;
(*d*) $x^2 = 2py$;
(*e*) $x^2 - y^2 = a^2$;
(*f*) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$;
(*g*) $(x - h)^2 + (y - k)^2 = r^2$;
(*h*) $(y - k)^2 = 2p(x - h)$;
(*i*) $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$.

Ans. (*a*) $-\dfrac{y_1}{x_1}$; (*b*) $3ax_1^2$; (*c*) $\dfrac{3ax_1^2}{2y_1}$;

(*d*) $\dfrac{x_1}{p}$; (*e*) $\dfrac{x_1}{y_1}$; (*f*) $\dfrac{b^2x_1}{a^2y_1}$;

(*g*) $-\dfrac{x_1 - h}{y_1 - k}$; (*h*) $\dfrac{p}{y_1 - k}$; (*i*) $-\dfrac{b^2(x_1 - h)}{a^2(y_1 - k)}$.

2. In each of the following curves find the slope of the tangent and of the normal at the point indicated:

(a) $9x^2 + y^2 = 18$, $(1, -3)$;
(b) $x^2 + 4y^2 = 4$, $(2, 0)$;
(c) $x^2 - y^2 = 25$, $(10, 5\sqrt{3})$;
(d) $5x^2 - 12y^2 = 13$, $(\sqrt{5}, -1)$;
(e) $x^2 + y^2 = 169$, $(5, -12)$;
(f) $x^2 + y^2 = 25$, $(3, 4)$;
(g) $2y^2 = 5x$, $(10, 5)$;
(h) $x^2 = 36y$, $(12, 4)$.

3. Write the equation of the tangent and normal to each of the curves in Problem 2 at the point indicated.

Ans. (a) Tangent, $3x - y - 6 = 0$; normal, $x + 3y + 8 = 0$.

4. Find the points at which the tangent to the corresponding curve in Problem 2 has the indicated slope:

(a) -3; (b) $-\frac{1}{2}\sqrt{3}$; (c) $-\frac{2}{3}\sqrt{3}$; (d) $\dfrac{\sqrt{5}}{12}$.
(e) $\frac{1}{2}$; (f) $\frac{1}{4}$; (g) $-\frac{1}{4}$; (h) $\frac{3}{2}$.

Ans. (a) $(1, 3)$, $(-1, -3)$.

5. Find the angle between each of the following pairs of curves. (By the angle between two curves is meant the angle between their tangents at the points of intersection.)

(a) $x^2 + y^2 = 25$, $x + y = 7$;
(b) $x^2 + y^2 = 12$, $y^2 = x$;
(c) $x^2 - y^2 = a^2$, $xy = \sqrt{2}\,a^2$;
(d) $x^2 + 4y^2 = 4$, $x + 2y = 2$.

Ans. (a) arc tan $\frac{1}{7}$.

6. Prove that the tangents at the ends of the latus rectum of a parabola are perpendicular to each other.

7. Find the angle between the tangents at the ends of a latus rectum of

(a) the ellipse; (b) the hyperbola.

Ans. (a) arc tan $\dfrac{2e}{1 - e^2}$.

8. Prove that the tangents at the ends of any chord through the focus of a parabola are perpendicular to each other.

9. An ellipse and hyperbola have the same foci. Prove that the tangents at their points of intersection are perpendicular to each other.

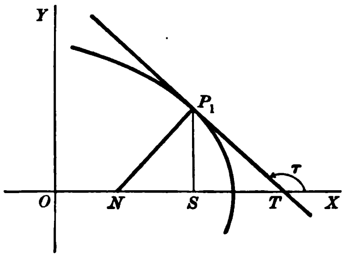
Hint. — The value of c is common to the curves.

10. Through what point of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ must a tangent and a normal be drawn so as to form with the principal axis an isosceles triangle?

Ans. $\left(\dfrac{\pm a^2}{\sqrt{a^2 + b^2}}, \dfrac{\pm b^2}{\sqrt{a^2 + b^2}}\right)$.

11. Prove that a circle described on a focal chord of a parabola as a diameter is tangent to the directrix.

138. Tangent and Normal. Subtangent and Subnormal. — Let P_1T be the tangent and P_1N the normal to the curve in the figure at P_1. By the *length of the tangent* is meant the distance from the point of tangency to the point where the tangent meets the x-axis, *i.e.* the length of P_1T. Similarly the length of the normal is the length of P_1N.



The *subtangent* is the projection of the tangent on the x-axis, *i.e.* TS. The *subnormal* is the projection of the normal on the x-axis, *i.e.* SN. The direction of reading the subtangent and subnormal is away from the intersection of the tangent with the x-axis; hence they are positive if they lie at the right of this point; negative if at the left.

To find the length of the subtangent and subnormal, let m be the slope and τ the inclination of the tangent. Then

$$m = \tan\tau = \frac{SP_1}{TS} = \frac{y_1}{TS}.$$

Hence
$$TS = \frac{y_1}{m}. \tag{47}$$

For the subnormal, the slope of NP_1 is $-\frac{1}{m}$. Then

$$-\frac{1}{m} = \tan P_1NS = \frac{SP_1}{NS} = -\frac{SP_1}{SN}.$$

Therefore
$$SN = my_1. \tag{48}$$

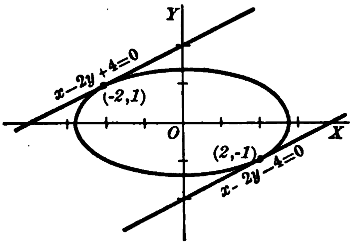
To find the lengths of the tangent and normal apply the right triangle theorem to the triangles NP_1S and SP_1T.

139. Tangent having a Given Slope. — Suppose it is required to find the tangent to the curve $x^2 + 4y^2 = 8$ having the slope $\frac{1}{2}$.

Here the slope is given and it is necessary to find the point of contact, which we call $P_1(x_1, y_1)$. Since P_1 is on the curve,

$$x_1^2 + 4y_1^2 = 8. \qquad (a)$$

Now $x^2 + 4y^2 = 8$ is an ellipse, for which $a^2 = 8$ and $b^2 = 2$. Hence by (44) the slope of the tangent at P_1 is $-\frac{2x_1}{8y_1}$. But by the conditions of the problem the slope is to be $\frac{1}{2}$. Hence



$$-\frac{2x_1}{8y_1} = \frac{1}{2}. \qquad (b)$$

Solving (a) and (b) simultaneously, we get $x_1 = \mp 2$, $y_1 = \pm 1$. Thus there are two tangents with points of contact $(2, -1)$ and $(-2, 1)$ and equations

$$y \pm 1 = \tfrac{1}{2}(x \mp 2),$$

which reduce to

$$x - 2y \mp 4 = 0.$$

The method illustrated in the above problem is typical and may be applied to any problem of the same character.

140. Tangent from a Given External Point. — To find the equation of a tangent to a curve from a given external point we proceed as in the following problem.

Let the curve be $x^2 + 4y^2 = 8$ and the point $(1, \frac{3}{2})$. As before, let the point of contact be $P_1(x_1, y_1)$. Then we have

$$x_1^2 + 4y_1^2 = 8. \qquad (a)$$

The slope of the tangent is, by (44), $-\frac{2x_1}{8y_1}$. Hence the equation of the tangent is

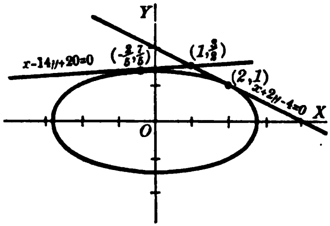
$$y - y_1 = -\frac{2x_1}{8y_1}(x - x_1).$$

Since this is to pass through $(1, \frac{3}{2})$, we have

$$\tfrac{3}{2} - y_1 = -\frac{2x_1}{8y_1}(1 - x_1). \qquad (b)$$

Solving equations (a) and (b) simultaneously, we get

$x_1 = 2$ or $-\frac{2}{5}$, $\quad y_1 = 1$ or $\frac{7}{5}$.



Thus there are two tangents with points of contact $(2, 1)$, and $(-\frac{2}{5}, \frac{7}{5})$. For the first of these the slope is $-\frac{1}{2}$ and the equation

$$y - 1 = -\tfrac{1}{2}(x - 2),$$

or
$$x + 2y - 4 = 0.$$

For the second the slope is $\frac{1}{14}$ and the equation

$$y - \tfrac{7}{5} = \tfrac{1}{14}(x + \tfrac{2}{5}),$$

or
$$x - 14y + 20 = 0.$$

PROBLEMS

1. Write the equations of the tangent and normal to each of the following curves at the point indicated:

(a) $x^2 + y^2 = 13$, $(2, -3)$; $\quad (e)$ $y = 2x^3$, $(2, 16)$;
(b) $x^2 + 2y^2 = 18$, $(4, -1)$; $\quad (f)$ $y^2 = -2x^3$, $(-2, -4)$;
(c) $x^2 - 2y^2 = 18$, $(-6, 3)$; $\quad (g)$ $y = 9x^2$, $(1, 9)$;
(d) $2y^2 + 5x = 0$, $(-10, 5)$; $\quad (h)$ $xy = 24$, $(4, 6)$.

2. Find the length of the subtangent and subnormal for each curve in Problem 1.

3. Find the equation of the tangent to each of the curves in Problem 1 parallel to the line $4x - 3y = 6$.

4. Find the equation of the tangent from the point $(5, -1)$ to each of the curves (a), (b), (c), and (d) in Problem 1.

5. Find the lengths of the subtangent, subnormal, tangent, and normal at any point (x, y) of

(a) $y^2 = 2px$; (c) $b^2x^2 - a^2y^2 = a^2b^2$;
(b) $b^2x^2 + a^2y^2 = a^2b^2$; (d) $x^2 + y^2 = r^2$.

Ans. Subtangent: (a) $2x$, (b) $-\frac{a^2y^2}{b^2x}$, (c) $\frac{a^2y^2}{b^2x}$, (d) $-\frac{y^2}{x}$; subnormal: (a) p, (b) $-\frac{b^2x}{a^2}$, (c) $\frac{b^2x}{a^2}$, (d) $-x$.

6. Write and simplify the equation of the normal to each of the curves of Problem 5 at any point (x_1, y_1).

7. Show that a line from the focus of the parabola $y^2 = 2px$ to the point where any tangent cuts the y-axis is perpendicular to the tangent.

8. Prove that the tangents at the ends of a latus rectum intersect on the directrix in the case of:

(a) the parabola; (b) the ellipse; (c) the hyperbola.

9. Prove that the tangents to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and the circle $x^2 + y^2 = a^2$ at points having the same abscissa meet on the x-axis.

10. Prove that the triangle formed by a tangent to the curve $2xy = a^2$, the x-axis, and a line joining the point of contact with the origin is isosceles.

11. Prove that the triangle formed by the tangent to the curve $2xy = a^2$ at any point and the coördinate axes has the area a^2.

12. A tangent to the parabola $y^2 = 2px$ has the intercepts on the axes numerically equal. Find its equation.

13. If a tangent to a parabola meets the latus rectum produced at A and the directrix at B, prove that $FA = FB$, F being the focus.

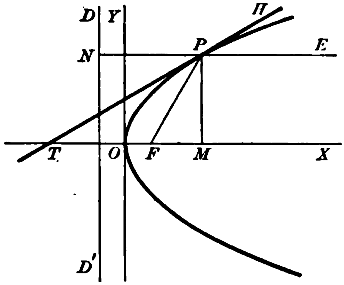
14. At what points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are the tangents equally inclined to both axes?

15. If d and d' are the distances of the foci of an ellipse from a tangent, prove that dd' is the square of the semi-minor axes.

141. The Parabolic Reflector. — Let TP be tangent to the parabola $y^2 = 2px$ at $P(x, y)$, and let TM be the subtangent.

Applying the subtangent formula, $TM = 2x$; hence, $TO = OM = x$. Now if F is the focus, this gives

$$TF = x + \frac{p}{2}.$$



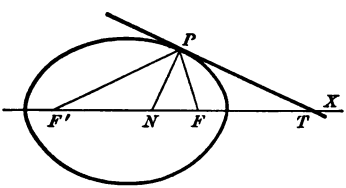
But $\quad FP = NP = x + \frac{p}{2}.$

Hence, TPF is isosceles, *i.e. the tangent to a parabola makes equal angles with the focal radius and the principal axis.*

This principle is used in the construction of parabolic reflectors. These are reflectors whose surfaces are generated by revolving a parabola about its principal axis. The lamp is placed at the focus. Then by the laws of optics a ray of light from F to any point on the surface P is reflected along PE so that angle TPF = angle HPE. But this makes angle HPE equal to angle PTF by the above proof. Therefore PE is parallel to OX. Thus all the rays of light from a lamp placed at F are reflected along lines parallel to the axis.

Exercise 5. Use the above property of the tangent to devise a method of constructing a tangent to a given parabola at any point by ruler and compasses.

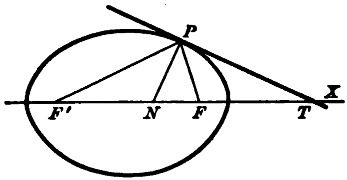
142. Angle between the Focal Radii and the Tangent to an Ellipse at any Point. — Let PT be a tangent and PN a normal to the ellipse at any point $P(x, y)$ on the ellipse, whose foci are F and F'. We desire to show that these lines make equal angles with the focal radii to P.



The slope of $F'P$ is $\frac{y}{x+c}$, of FP, $\frac{y}{x-c}$, and of NP, $\frac{a^2y}{b^2x}$, by § 135. Hence

$$\tan F'PN = \frac{\frac{a^2y}{b^2x} - \frac{y}{x+c}}{1 + \frac{a^2y^2}{b^2x(x+c)}} = \frac{a^2xy + a^2cy - b^2xy}{b^2x^2 + b^2cx + a^2y^2}$$

$$= \frac{c^2xy + a^2cy}{a^2b^2 + b^2cx} = \frac{cy}{b^2};$$



and

$$\tan FPN = \frac{\frac{y}{x-c} - \frac{a^2y}{b^2x}}{1 + \frac{a^2y^2}{b^2x(x-c)}} = \frac{b^2xy - a^2xy + a^2cy}{b^2x^2 - b^2cx + a^2y^2}$$

$$= \frac{a^2cy - c^2xy}{a^2b^2 - b^2cx} = \frac{cy}{b^2}.$$

Therefore $F'PN = FPN$; and the angles made with the tangent, being complementary to these, are also equal.

Exercise 6. Show how to draw a tangent to an ellipse at any point.

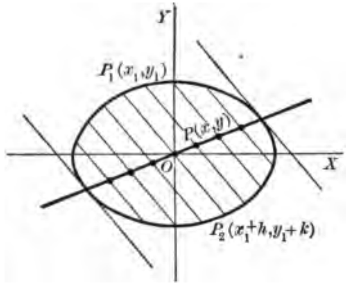
143. Diameters of a Conic. — The locus of the middle points of a system of parallel chords of any conic is called a *diameter* of the conic.

Let a chord with slope m meet the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the points $P_1(x_1, y_1)$ and $P_2(x_1+h, y_1+k)$. Then by § 135 the slope of the chord is

$$m = \frac{k}{h} = -\frac{b^2(2x_1+h)}{a^2(2y_1+k)}.$$



Now let $P(x, y)$ be the mid-point of the chord; then

$$2x = 2x_1 + h$$

and $$2y = 2y_1 + k$$

by the mid-point formulas.

Substituting, we have, $m = -\dfrac{b^2x}{a^2y}$ or

$$y = -\frac{b^2}{a^2m}x, \tag{49}$$

which is the equation of the locus. From this equation we see that the diameter is a straight line through the center of the ellipse of slope $m' = -\dfrac{b^2}{a^2m}$. Transferring the m to the other side, we have an important relation between the slopes of a diameter and its chords,

$$mm' = -\frac{b^2}{a^2}. \tag{50}$$

In similar manner, we may show that the equation of the diameter of the parabola $y^2 = 2px$, is

$$y = \frac{p}{m}, \tag{51}$$

where m is the slope of the system of chords.

For the hyperbola we obtain as the equation of the diameter of a set of chords of slope m,

$$y = \frac{b^2}{a^2m}x. \tag{52}$$

The relation between the slopes of the chords and their diameter is

$$mm' = \frac{b^2}{a^2}. \tag{53}$$

Exercise 7. Prove (51).

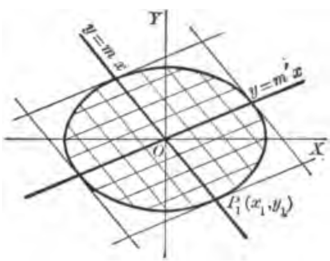
Exercise 8. Prove (52).

144. Conjugate Diameters of an Ellipse. — The relation

$$mm' = -\frac{b^2}{a^2}$$

states the condition satisfied by m and m' if the line $y = m'x$ is the diameter of a set of chords of slope m. But as m and m' can be interchanged in (50) without altering the relation, we see that $y = mx$ is the diameter of the set of chords of slope m', parallel to the first diameter.

Two diameters of slopes m and m', such that $mm' = -\frac{b^2}{a^2}$ are called *conjugate diameters.* Each diameter is one of the chords bisected by its conjugate.



It is readily seen that tangents at the ends of a diameter are parallel to the conjugate diameter.

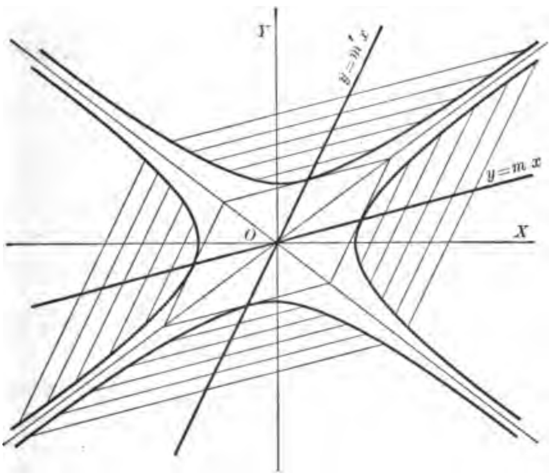
For, let $P_1(x_1, y_1)$ be the end of the diameter $y = mx$. The slope of the tangent at P_1 is $-\frac{b^2x_1}{a^2y_1}$ by § 135. But $\frac{x_1}{y_1} = \frac{1}{m}$, since P_1 lies on $y = mx$. Hence the slope of the tangent is $-\frac{b^2}{a^2m}$, which is also the slope of the conjugate diameter.

This property, together with Exercise 6, supplies a method of constructing a diameter conjugate to a given diameter with ruler and compasses.

145. Conjugate Diameters of the Hyperbola. — The relation between the slope of a diameter of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ and the slope of the corresponding chords has been found to be

$$mm' = \frac{b^2}{a^2}.$$

As the equation of the conjugate hyperbola, $-b^2x^2 + a^2y^2 = a^2b^2$, is obtained by substituting $-b^2$ for b^2 and a^2 for $-a^2$, it is easy to see that the corresponding relation for the diameter and chords of the conjugate hyperbola is $mm' = \frac{-b^2}{-a^2} = \frac{b^2}{a^2}.$ That is, it is the same in both the hyperbola and its conjugate. As in the case of the ellipse, two diameters of slopes m and m' such that $mm' = \frac{b^2}{a^2}$ are called *conjugate diameters.*



Similarly, the chords bisected by the diameter $y = mx$ are parallel to the conjugate diameter $y = m'x$. Also the tan-

gents to a hyperbola at the ends of a diameter are parallel to the conjugate diameter.

Exercise 9. Prove that the tangents at the ends of a diameter of a hyperbola are parallel to the conjugate diameter.

PROBLEMS

1. Find the diameter bisecting the system of chords of slope 2 in each of the following conics:

(*a*) $x^2 + 4y^2 = 16$; (*d*) $2y^2 - 5x = 0$;
(*b*) $x^2 - 9y^2 = 9$; (*e*) $9x^2 + 16y^2 = 144$;
(*c*) $x^2 + y^2 = 13$; (*f*) $16x^2 - 9y^2 = 144$.

2. Find the diameters conjugate to those obtained in Problem 1.

3. What is the relation holding between conjugate diameters of the ellipse $b^2y^2 + a^2x^2 = a^2b^2$?

4. Show that conjugate diameters of an ellipse lie in different quadrants.

5. Show that conjugate diameters of a hyperbola lie in the same quadrant.

6. Prove that the major axis of an ellipse is greater than any other diameter.

Hint. — Show that $4a^2$ is greater than the square of any diameter.

7. Lines are drawn joining the ends of the major and minor axes of an ellipse. Show that the diameters parallel to these are conjugate.

8. Prove that the tangents to an ellipse at the ends of a diameter make equal angles with lines joining these points to a focus.

9. Prove that the tangents to two conjugate hyperbolas at the ends of a pair of conjugate diameters meet on an asymptote.

10. Prove that if any pair of conjugate diameters of a hyperbola are equal, the hyperbola is equilateral.

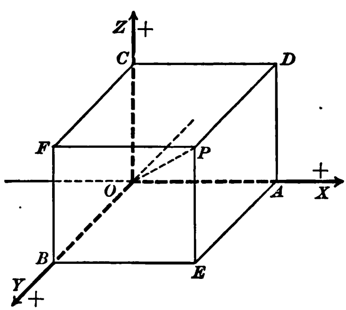
11. Prove that in an equilateral hyperbola the asymptotes bisect the angle between any pair of conjugate diameters.

CHAPTER XI

SOLID ANALYTIC GEOMETRY

146. Plane Analytic Geometry deals with the properties of figures which are wholly contained within a single plane. In Solid Analytic Geometry this restriction is removed. The effect of the addition of one dimension is to generalize the work of the previous chapters, the number of variables being increased to three, the loci of equations being surfaces, etc. The student should note carefully the correspondences between the two systems.

147. Coördinates. — The position of a point is determined with reference to three mutually perpendicular planes, XOY, YOZ, and XOZ. These are called the *coördinate planes*, and designated as the *xy-*, *yz-*, and *xz-planes* respectively. Their lines of intersection, OX, OY, and OZ, are called *coördinate axes* and designated as the *x-*, *y-*, and *z-axes* respectively; and their common point, O, is called the *origin*.

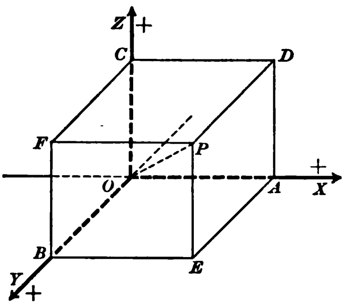


The *coördinates* of a point P are its distances from the respective coördinate planes measured parallel to the coördinate axes. The *x-coördinate* is the distance from the yz-plane parallel to the x-axis, the *y-coördinate* that from the xz-plane parallel to the y-axis, and the *z-coördinate* that from the

xy-plane parallel to the z-axis. For the point P in the figure,

$$x = FP = BE = CD = OA,$$
$$y = DP = AE = CF = OB,$$
$$z = EP = BF = AD = OC.$$

Corresponding to each point there are evidently always three coördinates. Conversely, a point is completely located by its coördinates. For the x-coördinate fixes it in a plane parallel to the yz-plane (or perpendicular to the x-axis), and similarly for the other coördinates; and these three planes meet in but one point. In plotting a point, we usually take



$$x = OA, \quad y = AE, \quad z = EP.$$

The positive directions are as indicated in the figure. The coördinate planes divide space into eight *octants*, which are distinguished by the signs of the coördinates of the points within them. For example the octant containing P in the figure has all the coördinates positive in sign.

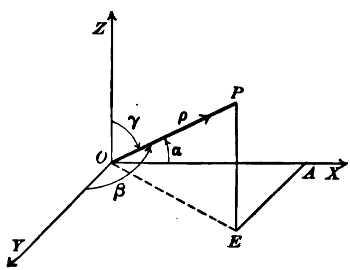
148. Radius Vector and Direction Cosines. — *The distance of a point from the origin is called its radius vector, and is denoted by* ρ.

For the point P in the figure $\rho = OP$. It is evident that

$$\rho^2 = \overline{OE}^2 + \overline{EP}^2$$
$$= \overline{OA}^2 + \overline{AE}^2 + \overline{EP}^2.$$

Hence

$$\rho^2 = x^2 + y^2 + z^2. \quad (54)$$



The angles between the line OP and the positive halves of the x-, y- and z-axes are called the *direction angles* of the line OP and are denoted by α, β, and γ, respectively. The cosines of these angles are called the *direction cosines* of the line. They evidently fix its direction. From the figure it is evident that

$$x = \rho \cos \alpha, \quad y = \rho \cos \beta, \quad z = \rho \cos \gamma. \tag{55}$$

Formulas (54) and (55) give at once the important relation

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \tag{56}$$

It is sometimes convenient to locate a point by means of its radius vector and its direction angles. In this case we write

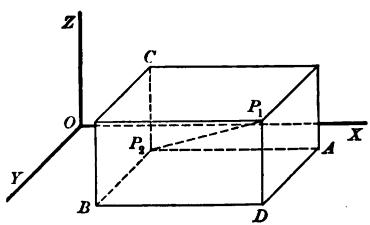
$$P(x, y, z) \equiv P(\rho, \alpha, \beta, \gamma).$$

This mode of representation is analogous to the polar coördinate system in plane analytic geometry.

149. Distance between Two Points. — *The distance between any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by the formula,*

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}. \tag{57}$$

To prove this, pass planes through the given points P_1 and P_2, parallel to the coördinate planes. These will form a rectangular parallelopiped, of which P_1P_2 is the diagonal.



By elementary geometry,

$$\overline{P_1P_2}^2 = \overline{P_2A}^2 + \overline{AD}^2 + \overline{DP_1}^2.$$

But $P_2A = x_1 - x_2$, $AD = y_1 - y_2$, and $DP_1 = z_1 - z_2$.

Hence we have the formula.

150. Direction of a Line. — *The direction angles of a line not passing through the origin are defined as the direction angles of a parallel line through the origin with the same positive direction.*

The positive direction on a line is arbitrary and indicated by the order in which its end-points are read. It is evident that if the direction is changed, the direction angles are replaced by their supplements. Thus, if the direction angles of P_2P_1 are α, β, and γ, those of P_1P_2 are $\pi - \alpha$, $\pi - \beta$, and $\pi - \gamma$. In all cases the direction angles are in value between 0 and π.

In the above figure the edges of the parallelopiped are parallel to the coördinate axes; hence the direction angles of the line P_2P_1 are:

$$\alpha = AP_2P_1,\ \beta = BP_2P_1,\ \gamma = CP_2P_1.$$

Then we have at once

$$\cos\alpha = \frac{x_1 - x_2}{d},\ \cos\beta = \frac{y_1 - y_2}{d},\ \cos\gamma = \frac{z_1 - z_2}{d}. \qquad (58)$$

Since the direction cosines are connected by formula (56), they may be found if three numbers to which they are proportional are known. For if

$$\cos\alpha : \cos\beta : \cos\gamma = a : b : c,$$

then

$$\frac{\cos^2\alpha}{a^2} = \frac{\cos^2\beta}{b^2} = \frac{\cos^2\gamma}{c^2} = \frac{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}{a^2 + b^2 + c^2} = \frac{1}{a^2 + b^2 + c^2},$$

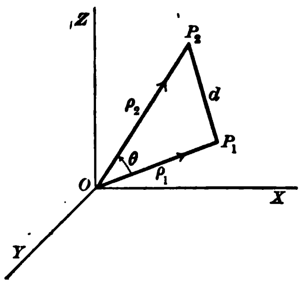
whence

$$\cos\alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}},\ \cos\beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}},$$

$$\cos\gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

Any three numbers proportional to the direction cosines of a line are called *direction numbers* of the line.

151. Angle between Two Lines. — *The angle between two lines which do not meet is defined as the angle between two intersecting lines parallel to the given lines and having the same positive directions.* If the lines are parallel, the angle between them is 0 or π, according to their directions.



We now derive a formula for the angle between two lines in terms of their direction cosines. Let the lines through the origin parallel to the given lines be OP_1 and OP_2, where the coördinates of P_1 and P_2 are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Let d be the distance between these points; let ρ_1 and ρ_2 be their radius vectors; and let θ be the angle between the lines. Then

$$\cos\theta = \frac{\rho_1^2 + \rho_2^2 - d^2}{2\rho_1\rho_2}.$$

But
$$\rho_1^2 = x_1^2 + y_1^2 + z_1^2,$$
$$\rho_2^2 = x_2^2 + y_2^2 + z_2^2,$$
$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.$$
$$\therefore \cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\rho_1\rho_2}.$$

Now from (55) $\frac{x_1}{\rho_1} = \cos\alpha_1$, etc. Hence this last equation becomes

$$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2. \quad (59)$$

When the two lines are parallel, we have

$$\alpha_1 = \alpha_2, \qquad \beta_1 = \beta_2, \qquad \text{and } \gamma_1 = \gamma_2,$$
or
$$\alpha_1 = \pi - \alpha_2, \qquad \beta_1 = \pi - \beta_2, \qquad \text{and } \gamma_1 = \pi - \gamma_2.$$

When the two lines are perpendicular, $\theta = \frac{\pi}{2}$ and (59) becomes

$$\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = 0. \quad (60)$$

PROBLEMS

1. Plot the points:

(a) (3, 4, 5), (6, 0, 0), (– 6, 1, 3);
(b) (– 2, 3, – 1), (7, – 4, 5), (1, 0, 1);
(c) (0, 0, 2), (0, 3, – 4), (– 2, – 2, 1);

2. For each of the above points find the radius vector and its direction cosines.

3. Generalize the definition of symmetry with respect to a line so that it will apply to symmetry with respect to a plane. Show that the point (x, y, z) is symmetrical to $(-x, y, z)$ with respect to the yz-plane; to $(x, -y, z)$ with respect to the xz-plane; to $(x, y, -z)$ with respect to the xy-plane.

4. Show that the point (x, y, z) is symmetrical to $(-x, -y, z)$ with respect to the z-axis; to $(-x, y, -z)$ with respect to the y-axis; to $(x, -y, -z)$ with respect to the x-axis.

5. Show that the following points are vertices of a regular tetrahedron:

(a) (4, 0, 0), (0, 4, 0), (0, 0, 4), (4, 4, 4);
(b) (6, – 2, 1), (3, 1, 1), (3, – 2, 4), (2, – 3, 0).

6. Show that the three points (1, 4, – 2), (4, 7, 1), (– 2, 1, – 5) are in a straight line.

7. Show by two methods that the following are the vertices of a right triangle:

(a) (– 2, 3, 5), (– 5, – 3, 3), (0, 0, 11);
(b) (2, 6, – 5), (– 1, 0, – 7), (4, 3, 1).

8. Find the angles of the triangles whose vertices are given in Problem 1.

9. Find the coördinates of the point determined as follows:

$$\rho = 8,\ \cos\alpha = \tfrac{1}{2},\ \cos\gamma = -\tfrac{1}{3}.$$

10. Show that the coördinates of the point P_0 dividing the line P_1P_2 in the ratio $r_1 : r_2$ are

$$x_0 = \frac{r_2x_1 + r_1x_2}{r_1 + r_2},\quad y_0 = \frac{r_2y_1 + r_1y_2}{r_1 + r_2},\quad z_0 = \frac{r_2z_1 + r_1z_2}{r_1 + r_2}.$$

11. What is the locus of points for which

(a) $z = 0$;
(b) $y =$ a constant;
(c) $y = x = 0$;
(d) $x = y$;
(e) $x = 3,\ z = -2$;
(f) $\rho = 6$;
(g) $\cos\alpha = 0$;
(h) $\cos\alpha = \cos\beta = 0$?

12. What is the projection of $P(-2, 3, -4)$ on

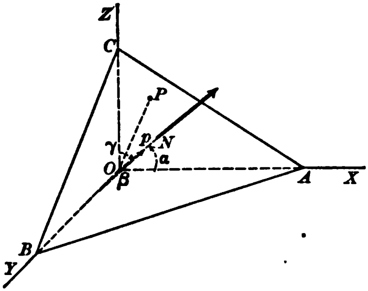
(a) the xy-plane;
(b) the z-axis?

152. The Locus in Solid Geometry. — *The locus of a point in space satisfying a given set of conditions is in general a surface or group of surfaces.* To illustrate, consider the locus of a point at a given distance a from a given fixed point C. This is evidently the definition of the surface of a sphere of radius a and center C. Again the locus of a point at a constant distance a from a given straight line l is evidently a circular cylindrical surface having l as an axis and a as a radius.

Extending the definition of the equation of a locus given in Chapter I, this shows that the locus of a single equation in the three variables, x, y, and z, is in general a surface.

If the coördinates of a point satisfy *two* equations simultaneously, its locus must be common to the two surfaces which are the loci of the equations, *i.e.* it must be their intersection, which is a curve or set of curves. Finally the points determined by a set of *three* equations will be the intersections of the surface determined by the third with the curves determined by the first two, and hence will be isolated points. Note the analogy to plane analytic geometry, where *one* equation determines a curve, and *two* equations the isolated points in which the curves meet.

153. The Normal Equation of the Plane. — The perpendicular upon a plane from the origin is known as the *normal axis* and the distance of the plane from the origin is known as the *normal intercept*. A plane is completely determined if the length of the normal intercept p, and the direction angles, α, β, and γ, of the normal axis are known.



To find the normal equation of a plane, take any point $P(x, y, z)$ in the plane and join it to the origin. Draw the normal axis ON. Let ρ_1 be the radius vector of P, and α_1, β_1, γ_1, its direction angles; let α, β, γ be the direction angles of ON; and let θ be the angle between OP and ON.

Then by (59),

$$\cos\theta = \cos\alpha_1\cos\alpha + \cos\beta_1\cos\beta + \cos\gamma_1\cos\gamma.$$

But $\cos\theta = \dfrac{ON}{OP} = \dfrac{p}{\rho_1}$.

Eliminating θ between these equations,

$$\rho_1[\cos\alpha_1\cos\alpha + \cos\beta_1\cos\beta + \cos\gamma_1\cos\gamma] = p.$$

This becomes on applying (55),

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p, \tag{61}$$

which is the *normal equation.*

Exercise. — Show that for any point $P(x, y, z)$ not in the plane

$$x\cos\alpha + y\cos\beta + z\cos\gamma \gtrless p.$$

154. Plane Parallel to One or More Coördinate Axes. — Suppose the plane is given parallel to the z-axis. In this case the normal axis will lie in the xy-plane and $\gamma = \dfrac{\pi}{2}$. Hence (61) reduces to the form

$$x\cos\alpha + y\cos\beta = p. \tag{61a}$$

Similar equations are obtained for planes parallel to either of the other axes.

In case the plane is parallel to both the y- and z-axes, we have the normal along the x-axis. Then $\beta = \gamma = \frac{\pi}{2}$, and $\alpha = 0$. For this case (61) reduces to the form

$$x = p,$$

a result obvious from the definition of coördinates.

155. The General Equation of the First Degree. — The normal equation (61) of the plane is of the first degree. We now wish to prove conversely that the locus of the general first-degree equation

$$Ax + By + Cz + D = 0 \tag{62}$$

is a plane. To do this we simply show that (62) can always be reduced to the form (61).

Dividing both sides of the equation by $\pm\sqrt{A^2 + B^2 + C^2}$, we have

$$\frac{A}{\pm\sqrt{A^2 + B^2 + C^2}}x + \frac{B}{\pm\sqrt{A^2 + B^2 + C^2}}y + \frac{C}{\pm\sqrt{A^2 + B^2 + C^2}}z = \frac{-D}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

By § 150 the coefficients of x, y, and z are direction cosines of a line. Hence this equation has the same form as (61) and its locus is a plane. The direction cosines of its normal are

$$\frac{A}{\pm\sqrt{A^2 + B^2 + C^2}}, \quad \frac{B}{\pm\sqrt{A^2 + B^2 + C^2}}, \quad \frac{C}{\pm\sqrt{A^2 + B^2 + C^2}}$$

and the length of its normal intercept is

$$\frac{-D}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

The sign of the radical is taken opposite to that of D so that the normal distance p shall be positive. Comparison with (61a) shows that if any one of the three variables is missing the locus is a plane parallel to the corresponding axis.

156. Angle between Two Planes. — The angle between two planes is readily seen to be the same as that between their normal axes. Hence the angle between two planes can be found by formula (59). If we wish to express $\cos\theta$ in terms of the coefficients of (62), substitution of the above values of the direction cosines gives

$$\cos\theta = \frac{AA' + BB' + CC'}{\sqrt{A^2 + B^2 + C^2} \cdot \sqrt{A'^2 + B'^2 + C'^2}}. \tag{63}$$

Parallel Planes. — If two planes are parallel, their normal axes are the same. Hence their direction cosines are the same or numerically equal with unlike signs. This gives

$$\frac{A}{\sqrt{A^2+B^2+C^2}} = \frac{\pm A'}{\sqrt{A'^2+B'^2+C'^2}}, \quad \frac{B}{\sqrt{A^2+B^2+C^2}} = \text{etc.}$$

By alternation we have

$$\frac{A}{A'} = \frac{\sqrt{A^2+B^2+C^2}}{\sqrt{A'^2+B'^2+C'^2}}, \quad \frac{B}{B'} = \frac{\sqrt{A^2+B^2+C^2}}{\sqrt{A'^2+B'^2+C'^2}},$$

$$\frac{C}{C'} = \frac{\sqrt{A^2+B^2+C^2}}{\sqrt{A'^2+B'^2+C'^2}}.$$

This gives as the condition for parallelism,

$$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}. \tag{64}$$

Perpendicular Planes. — If the planes are perpendicular, $\cos\theta = 0$. Then from (63) we have at once

$$AA' + BB' + CC' = 0. \tag{65}$$

157. The Intercept Equation. — The *intercepts* of a plane are the distances from the origin to the points in which it meets the x-, y-, and z-axes. They are denoted by a, b, and c, respectively.

If in equation (62) we set $y = z = 0$, we find that the

x-intercept, a, is $-\frac{D}{A}$. Similarly $b = -\frac{D}{B}$, and $c = -\frac{D}{C}$. By transposing D and dividing both sides by $-D$, (62) may be reduced at once to the form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \tag{66}$$

which is known as the *intercept form* of the equation.

PROBLEMS

1. Write the equations of the following planes in the intercept and normal forms and determine which octant contains the foot of the normal axis.

(*a*) $3x - 4y + 8z - 5 = 0$;
(*b*) $4x + y + z - 8 = 0$;
(*c*) $3x - y - 8z + 10 = 0$;
(*d*) $4x + 5y - 20 = 0$;
(*e*) $x + z - 1 = 0$;
(*f*) $x + y + z = 12$.

2. Write the equations of the planes determined by the following data:

(*a*) $a = 3$, $b = 4$, $c = -\frac{1}{2}$;
(*b*) $p = 5$, $\cos\alpha = \frac{1}{2}$, $\cos\beta = -\frac{2}{3}$;
(*c*) through the points (1, 1, 1), (2, -1, -3), (4, 3, -2);
(*d*) parallel to the plane $3x - 4y + 8z - 5 = 0$ and containing the point (2, 1, -4);
(*e*) perpendicular to the plane $3x - 4y + 8z - 5 = 0$ and containing the points (2, 1, -4) and the origin;
(*f*) having the foot of the normal axis at the point (-2, -4, 1);
(*g*) parallel to the z-axis and containing the points (2, -4, 1) and (3, 5, -2).

3. Find the angle between each of the planes of Problem 1 and the plane $8x + 2y + 2z - 5 = 0$.

4. Find the angles at which each of the planes of Problem 1 meets the coördinate planes.

5. Show that the distance from the plane $x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0$ to the point (x_1, y_1, z_1) is $x_1\cos\alpha + y_1\cos\beta + z_1\cos\gamma - p$.

6. Using the result of Problem 5 find the distance from each of the planes of Problem 1 to the point (3, 0, -2).

7. Find the locus of points equidistant from the points (-2, -3, 1) and (4, 5, -7).

8. Find the point of intersection of the planes

$$\begin{aligned} 2x - 3y + 2z &= 5;\\ 8x + 5y - 3z &= 6;\\ 4x - 2y + z &= 9. \end{aligned}$$

9. Show that the planes bisecting and perpendicular to the edges of the tetrahedron whose vertices are $(3, 1, -2)$, $(4, 2, -1)$, $(6, 1, -5)$, $(1, 3, 2)$ meet in a point.

10. Find the locus of points equidistant from the planes $2x - 3y + z = 4$ and $3x - 2y + z = 5$.

11. Find the volume of the tetrahedron whose vertices are given in Problem 9.

12. What is the locus of each of the following equations:

$$\begin{aligned} &(a)\ x^2 - 3x + 2 = 0;\\ &(b)\ x^2 - y^2 = 0;\\ &(c)\ x^2 + 2xy + y^2 - z^2 = 0? \end{aligned}$$

13. Name two of the following planes which are (*a*) parallel; (*b*) perpendicular to each other:

$$\begin{aligned} 2x - 3y + 5z &= 8;\\ 4x + 6y + 2z &= 1;\\ 6x - 9y + 15z &= 5. \end{aligned}$$

158. The Equations of the Line. — Since any straight line may be regarded as the intersection of two planes, it will be seen from §§ 152 and 155 that it requires *two* equations of the form

$$Ax + By + Cz + D = 0$$

to determine a straight line.

It is more convenient, however, to determine a line from two given points upon it, or from one point and its direction angles. From these we derive equations for the line. It should be noted that in each case *two* equations are required.

159. The Symmetrical Equations. — Let the line pass through the point $P_1(x_1, y_1, z_1)$ and have direction angles α, β, and γ. If $P(x, y, z)$ is any point on the line and d denotes the distance P_1P, we have from (58)

$$\cos\alpha = \frac{x - x_1}{d},\ \cos\beta = \frac{y - y_1}{d},\ \cos\gamma = \frac{z - z_1}{d}.$$

These relations give,

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}, \tag{67}$$

which are the *symmetrical equations.*

If a point P_1 and the direction numbers of the line a, b, and c are known, a more convenient form to use is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}. \tag{67a}$$

160. The Two-point Equations. — Let the line be determined by the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2\ y_2\ z_2)$. Then from (58) the difference of the respective coördinates are direction numbers. Hence the two-point equations may be obtained from (67a) by substituting $x_1 - x_2$, $y_1 - y_2$, and $z_1 - z_2$, for a, b, and c, respectively. They are

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}. \tag{67b}$$

161. The Projection Forms. — Let the equations of the projections of the given line on the xy- and xz-planes respectively be

$$\begin{aligned} y &= kx + m, \\ z &= lx + n. \end{aligned} \tag{68}$$

These equations taken together determine the line.

For each equation may be regarded as the equation of the projecting plane. For example, the first by (61a) is the equation of a plane parallel to the z-axis and hence perpendicular to the xy-plane. But it contains all points satisfying the relation $y = kx + m$ and so must contain the given projection. Similarly, the other equation is that of the plane projecting the line on the xz-plane. Thus the line is determined as the intersection of its projecting planes.

162. Reduction of the General Equations of a Line to the Standard Forms. — Consider the line determined by the equations

$$4x + y - z + 2 = 0,$$
$$x + 4y + 2z - 1 = 0.$$

To reduce these to the projection forms, eliminate first z and then y between the equations. We have at once,

$$3x + 2y + 1 = 0, \quad 5x - 2z + 3 = 0.$$

That the locus of these equations is the same as the locus of those given is obvious, since coördinates satisfying the first pair of equations must satisfy the last.

To reduce the given equations to the symmetrical form solve the projection forms for x. We have,

$$x = \frac{2y+1}{-3}, \quad x = \frac{2z-3}{5}.$$

Equating these, we have

$$\frac{x}{1} = \frac{y + \frac{1}{2}}{-\frac{3}{2}} = \frac{z - \frac{3}{2}}{\frac{5}{2}}.$$

The denominators are direction numbers; and if we divide each one by $\sqrt{1^2 + (-\frac{3}{2})^2 + (\frac{5}{2})^2} = \sqrt{\frac{38}{4}}$, they become direction cosines by § 150. Doing so, we get

$$\frac{x}{\frac{2}{\sqrt{38}}} = \frac{y + \frac{1}{2}}{\frac{-3}{\sqrt{38}}} = \frac{z - \frac{3}{2}}{\frac{5}{\sqrt{38}}}.$$

which are the symmetrical equations of a line having direction cosines $\frac{2}{\sqrt{38}}, \frac{-3}{\sqrt{38}}, \frac{5}{\sqrt{38}}$, and passing through the point $(0, -\frac{1}{2}, \frac{3}{2})$. But these are merely transformations of the original equations, and so are the symmetrical forms required.

PROBLEMS

1. Find the points at which the following lines cut the coördinate planes and draw the lines:

(*a*) $2x - 3y + 2z - 5 = 0,\ 3x - y - z + 6 = 0$;
(*b*) $2x - 4y = 5,\ 3y - z = 2$;
(*c*) $\dfrac{x-2}{6} = \dfrac{y-3}{2} = \dfrac{z+1}{3}$.

2. Reduce the equations in Problem 1 to the symmetrical form.

3. Reduce the equations in Problem 1 to the projection form, the projecting planes being perpendicular to the xz- and yz-planes.

4. Find the equations of the straight lines determined as follows:
(*a*) through the points $(1, 0, -5)$ and $(-2, 3, 1)$;
(*b*) through the point $(1, -1, 2)$ and parallel to the z-axis;
(*c*) through the point $(1, -1, 2)$ and perpendicular to the z-axis;
(*d*) through the point $(1, -1, 2)$ and having $\cos\alpha = \frac{1}{2}$, $\cos\beta = \frac{1}{4}$.
(*e*) through the point $(1, -1, 2)$ and parallel to the line of Problem 1 (*a*).

5. Do the lines of Problem 1 (*a*) and (*b*) meet?

6. Show that if a line is perpendicular to a plane, it has the same direction cosines as the normal to the plane.

7. Show that if a line is parallel to a plane its direction cosines and those of the normal to the plane satisfy the relation

$$\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' = 0.$$

8. Find the angle between the lines

$$\frac{x-3}{2} = \frac{y+2}{3} = \frac{z-5}{6} \text{ and } \frac{x+3}{6} = \frac{y-2}{3} = \frac{z-4}{1}.$$

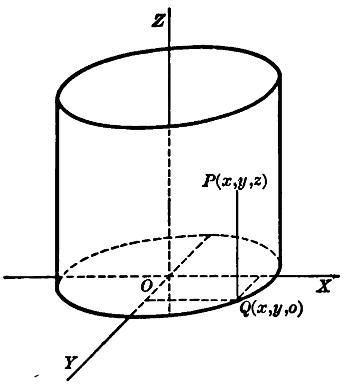
9. Show that the line $\dfrac{x-3}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{1}$

(*a*) is parallel to the plane $6x - 8y + 12z - 60 = 0$;
(*b*) lies in the plane $3x - 4y + 6z - 19 = 0$;
(*c*) is perpendicular to the plane $2x + 3y + z = 6$.

10. Find in the symmetrical form the equations of the locus of points
(*a*) equidistant from the points $(3, -1, 2)$, $(4, -6, -5)$, and $(0, 0, -3)$;
(*b*) equidistant from the planes $2x - 3y = 6$, $3x - 2z = 8$, and $2x + 3z = 5$.

163. Cylindrical Surfaces. — *A cylindrical surface is a surface generated by a moving straight line which constantly intersects a given fixed curve and remains parallel to a fixed straight line.* The fixed curve is called the *directrix* and the generating line the *generatrix.*

Let the directrix of a cylindrical surface be an ellipse in the xy-plane symmetrical with respect to the coördinate axes, and let the generatrix remain parallel to the z-axis. Then the equation of the ellipse in its plane will be



$$b^2x^2 + a^2y^2 = a^2b^2.$$

Let $P(x, y, z)$ be any point on the surface, and Q the corresponding point on the ellipse. Evidently the coördinates of Q will be $(x, y, 0)$; *i.e.* the x- and y-coördinates of every point on the line PQ are the same as those of Q. But the coördinates of Q satisfy the equation of the ellipse. Hence for any point on the cylindrical surface,

$$b^2x^2 + a^2y^2 = a^2b^2.$$

The preceding is perfectly general, and we see that it leads at once to the theorem:

The locus of a cylindrical surface which has for its directrix a curve in the xy-plane and whose generatrix moves parallel to the z-axis, has the same equation as the directrix.

The converse, which is readily established, is:

The locus of an equation in two variables, x and y, is a cylindrical surface, whose directrix is the curve in the xy-plane which has the same equation, and whose generatrix moves parallel to the z-axis.

Similar theorems hold of course for cylinders whose generatrices are parallel to either the x- or y-axes.

Exercise. What is the locus of any equation in *one* variable?

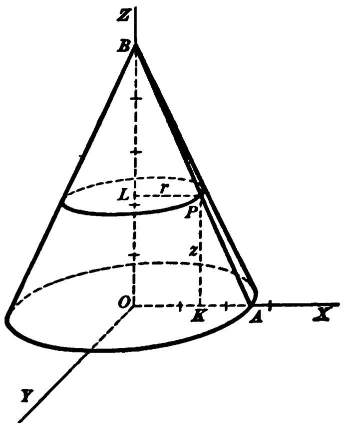
164. Surfaces of Revolution. — *A surface generated by revolving a plane curve about a fixed line in its plane as an axis is called a surface of revolution.*

The curve is called the *generatrix;* its position with reference to the axis is unchanged during the revolution. Sections of the surface made by planes through the axis are called *meridional sections.* From the definition of such a surface it is evident that

(*a*) sections made by planes perpendicular to the axis are circles;

(*b*) any meridional section is the generatrix itself.

Let us first consider a conical surface generated by revolving a straight line about the z-axis. Let the position of the generatrix in the xz-plane be AB, of which the equation is



$$2x + z = 5.$$

Let $P(x, y, z)$ be any point of the locus. As the line turns about the z-axis, P describes a circle of radius $r = LP$ in a plane parallel to the xy-plane and distant from it $KP = z$. Now when the line is in the xz-plane, $x = r$; hence for all positions of the line,

$$2r + z = 5.$$

But, as P describes a circle of radius r in a plane parallel to the xy-plane, we have at once

$$x^2 + y^2 = r^2.$$

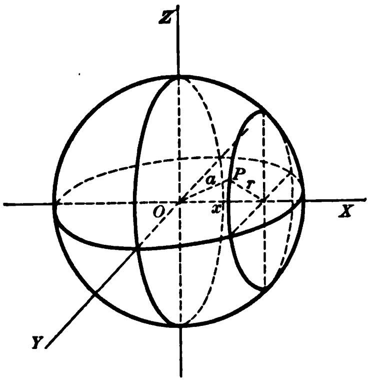
Substituting above, we have

$$2\sqrt{x^2+y^2}+z=5.$$

Simplifying, we have the equation

$$4(x^2+y^2)=(z-5)^2.$$

Consider the surface generated by revolving about the x-axis a circle of radius a whose center is the origin. The equation of the generatrix in the xz-plane is



$$x^2+z^2=a^2.$$

Evidently the point P describes a circle with its center on the x-axis and of radius r; hence

$$x^2+r^2=a^2.$$

But $y^2+z^2=r^2$.

Hence we have at once,

$$x^2+y^2+z^2=a^2. \qquad (69)$$

This equation is important, as it is the equation of a sphere of radius a and having its center at the origin.

PROBLEMS

1. Identify and sketch the following surfaces:

(a) $x^2-4y^2=4$;
(b) $x^2+2z^2=2$;
(c) $y^2+z^2+2y+6z=15$;
(d) $xz=15$;
(e) $y^2+4y-5=0$;
(f) $x^2=16z$.

2. Find the equation of the surface generated by revolving the curve about the axis indicated:

(a) $3x+12y=3$, x-axis;
(b) $z^2=2px$, x-axis;
(c) $b^2x^2+a^2z^2=a^2b^2$, x-axis;
(d) $b^2x^2+a^2z^2=a^2b^2$, z-axis;
(e) $b^2x^2-a^2z^2=a^2b^2$, x-axis;
(f) $b^2x^2-a^2z^2=a^2b^2$, z-axis.

3. Find the equation of the surface of a cone whose vertex is at the origin, with the z-axis for its axis of revolution and opposite elements perpendicular to each other.

4. Find the equation of the locus of a point equidistant from a given plane and a given line parallel to the plane.

5. Show that the locus of a point equidistant from the point $(p, 0, 0)$ and the yz-plane is the paraboloid of revolution $y^2 + z^2 = 2px - p^2$.

6. Show that the locus of a point in space the sum of whose distances from the points $(\pm c, 0, 0)$ is $2a$ is the prolate spheroid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$.

7. How may the hyperboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ be defined as a locus?

165. Discussion of Surfaces. — The discussion of a surface is in general a much more complicated matter than that of a curve. The notions of intercepts, symmetry, and extent are readily generalized. Besides these, however, we have two other notions, which are of help in determining the nature of a surface. The first is the section of the surface by any plane parallel to one of the coördinate planes; the second is the section of the surface by any of the coördinate planes. The latter is called a *trace* of the surface.

To illustrate these, consider the sphere, with equation

$$x^2 + y^2 + z^2 = a^2.$$

To find the equations of the traces, set x, y, and z successively equal to 0. When $x = 0$, we have

$$y^2 + z^2 = a^2.$$

Thus the yz-trace is a circle of radius a. The xy- and xz-traces are also circles of the same radius, with equations

$$x^2 + y^2 = a^2$$

and $$x^2 + z^2 = a^2 \text{ respectively.}$$

The section made by a plane parallel to the yz-plane and at a distance c from it is the curve determined by the given equation and the equation $x = c$. Eliminating x between the two equations, we have

$$y^2 + z^2 = a^2 - c^2,$$

which shows that the section is a circle having the point $(c, 0, 0)$ as a center and $\sqrt{a^2 - c^2}$ as the radius. If $c > a$, this is imaginary, indicating that the surface lies wholly within the planes $x = \pm a$. A similar discussion holds for sections parallel to the other coördinate planes.

To sketch a surface when only two variables of its equation are of the same degree, or of the same degree and sign, first draw sections parallel to the plane of the two variables. These sections and the traces give the best representation of the locus.

166. Quadric Surfaces. — By analogy to the conics, the locus of the general equation of the second degree

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Kz + L = 0$$

is called a *quadric surface.* It can be shown that these surfaces have properties analogous to those of the conics; and in particular that every plane section of such a surface is a conic. We shall not go into these details, but will confine our discussion to the simpler forms to which the general equation may be reduced.

167. The Ellipsoid. — The locus of the equation

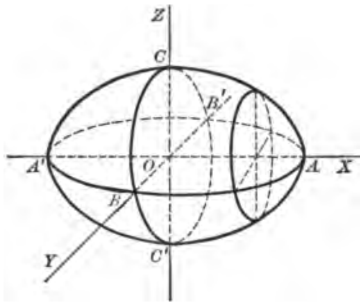
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{70}$$

is called an ellipsoid.

From the form of the equation we see at once that the

surface is symmetrical with respect to all of the coördinate planes and coördinate axes. The intercepts on the axes are

$$x = \pm a,$$
$$y = \pm b,$$
$$z = \pm c.$$



Setting $x = 0$, the equation of the trace on the yz-plane is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

hence it is an ellipse with semi-axes b and c. In like manner the traces on the other coördinate planes are ellipses.

Setting $x = k$, the equation becomes $\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{a^2 - k^2}{a^2}$, a form which shows that there is no section for k numerically greater than a. Dividing by the right-hand member, this becomes

$$\frac{y^2}{\frac{b^2}{a^2}(a^2 - k^2)} + \frac{z^2}{\frac{c^2}{a^2}(a^2 - k^2)} = 1.$$

Hence the various sections are ellipses symmetrical with respect to the x-axis. The semi-axes, $\frac{b}{a}\sqrt{a^2 - k^2}$ and $\frac{c}{a}\sqrt{a^2 - k^2}$, grow smaller as k increases, until the ellipse becomes a point when $k = a$. Similarly the sections parallel to the other coördinate planes are ellipses.

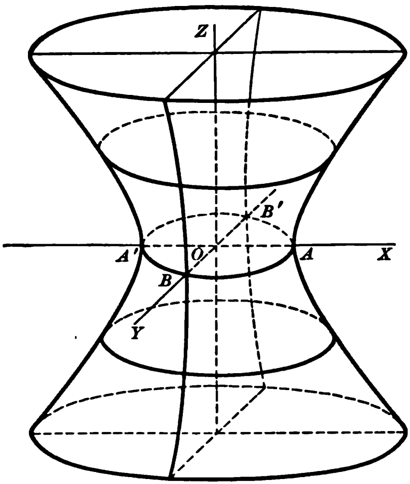
The values a, b, and c are called the semi-axes of the ellipsoid. When $a = b = c$, the ellipsoid is evidently a sphere with the center at the origin. Ordinarily the semi-axes are unequal, and in this form of the equation it is usually assumed that $a > b > c$. When $b = c$, but $a > b$ or c, the

ellipsoid is called a prolate spheroid. In this case the sections parallel to the yz-plane are circles. The locus is then a surface of revolution since it is generated by revolving the xz-trace (or the xy-trace) about the x-axis. When $a = b$ but $c < a$ or b, the surface is that of an oblate spheroid. This is also a surface of revolution and the sections parallel to the xy-plane are circles.

168. Hyperboloid of One Sheet. — The locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{71}$$

is called a hyperboloid of one sheet.



In this case sections parallel to the xy-plane, made by the planes $z = k$, have equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}$$

and hence are ellipses which increase in size as the numerical value of k increases.

The xz-trace and the yz-trace are the hyperbolas

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1 \text{ and } \frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \text{ respectively.}$$

If $a=b$, the locus is a surface of revolution about the z-axis.

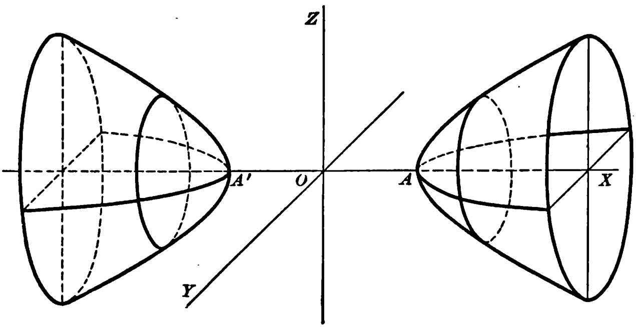
169. Hyperboloid of Two Sheets. — The locus of the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad (72)$$

is called the hyperboloid of two sheets.

In this case we consider sections parallel to the yz-plane made by the planes $x=k$. These are the ellipses

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{k^2}{a^2}-1.$$



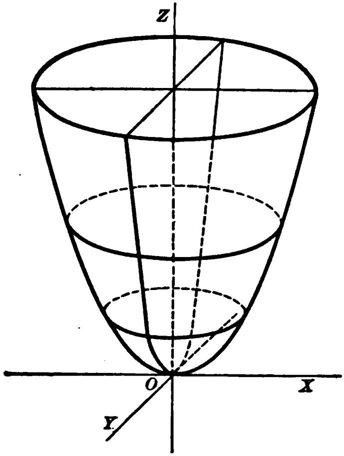
If $k^2 < a^2$ the right-hand member is negative. Hence no part of the locus is between the two planes $x=\pm a$. If $k=\pm a$, each corresponding section is a point ellipse. As k increases numerically from a to ∞ the section increases indefinitely.

The xy-trace and the xz-trace are the hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ and } \frac{x^2}{a^2}-\frac{z^2}{c^2}=1 \text{ respectively.}$$

If $b=c$, the locus is a surface of revolution.

170. Elliptic Paraboloid. — This is the locus of the equation



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\,cz. \tag{73}$$

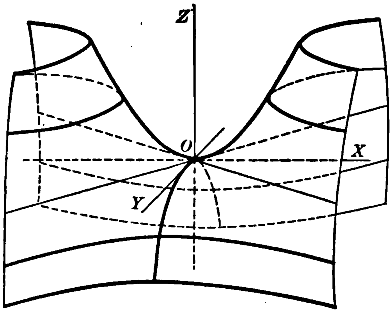
Proceeding as before, we observe that sections made by the planes $z=k$ are ellipses whose axes increase indefinitely as k increases. The surface lies wholly above or wholly below the xy-plane, according as c is positive or negative. The xz- and yz-traces are parabolas. If $a=b$, the locus is a surface of revolution.

171. Hyperbolic Paraboloid. — This is the locus of the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2\,cz. \tag{74}$$

Consider c positive. Sections made by the planes $z=k$ are the hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2\,ck.$$



As k increases from 0 to ∞ the vertices of the corresponding sections lie in the xz-plane and recede indefinitely from the z-axis. As k decreases from 0 to $-\infty$ the vertices of the sections are in the yz-plane and recede indefinitely from the -xis. The xz- and yz-traces are parabolas, each having its

vertex at the origin, the former extending above the xy-plane, the latter below. The xy-trace is a pair of lines intersecting at the origin.

PROBLEMS

1. Find the traces of the following surfaces on each coördinate plane:

(*a*) $y^2 + z^2 = 12x$;
(*b*) $x^2 - y^2 = 0$;
(*c*) $x^2 + 4y^2 + 4z^2 = 4$;
(*d*) $x^2 + 4y^2 - 4z^2 = 4$;
(*e*) $x^2 - 4y^2 - 4z^2 = 4$;
(*f*) $x^2 + y^2 = 9z^2$;
(*g*) $x^2 + 4y^2 + 2z^2 = 16$;
(*h*) $x^2 = y^2 + 9z^2$.

2. Find the intersection of each of the above surfaces and the plane $x = 4$.

3. Discuss the traces of the following surfaces:

(*a*) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$;

(*b*) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = z$;

(*c*) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = z$.

4. Show that the sections of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = z$ are parabolas if perpendicular to the x- or the y-axis, and ellipses if perpendicular to the z-axis and $z > 0$.

5. Show that the sections of the surface $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = z$ perpendicular to the x- or the y-axis are parabolas. How do those perpendicular to the x-axis differ from those perpendicular to the y-axis?

6. Discuss and sketch the following surfaces:

(*a*) $4x^2 + 25y^2 + 16z^2 = 100$;
(*b*) $28x^2 + 196 = 9y^2 + 16z^2$;
(*c*) $4x^2 + 9y^2 - z^2 + 36 = 0$;
(*d*) $16x^2 + y^2 + 64z = 0$;
(*e*) $100z^2 + 36y^2 - 81x = 0$;
(*f*) $y^2 - z^2 + 9x = 0$.

FORMULAS AND EQUATIONS

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3 86 [illegible]

C. X. [illegible]

$$\sqrt{\left(x-\frac{p}{2}\right)^2+y^2}=\sqrt{x^2}$$

$$\left(x-\frac{p}{2}\right)^2+y^2=x^2$$

$$\left(x-\frac{p}{2}\right)^2+y^2=\cancel{x^2}$$

$$\cancel{x^2}-xp+\frac{p^2}{4}+y^2$$

$$y^2=xp-\frac{p^2}{4}$$

$$4xp-p^2$$

$$4y^2=4xp-p^2$$

$$y^2=$$